FORCED OSCILLATIONS IN NONLINEAR FEEDBACK CONTROL SYSTEM

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FORCED OSCILLATIONS IN NONLINEAR FEEDBACK CONTROL SYSTEM

by

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Lieutenant Commander, Chinese Navy

Submitted in partial fulfillment of
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ABSTRACT

Since a nonlinear feedback control system may possess more than one type of forced oscillations, it is highly desirable to investigate the type of forced oscillations which can occur when the nonlinear restoring force function is of a specific form.

This paper is to present a harmonic linearization method for finding the existence of forced oscillations and response curve characteristics of a nonlinear feedback control system by means of finding the restoring force function of the nonlinearity and using the harmonic balance and an iteration method for investigating the conditions for one type of forced oscillations exhibited.

The existence conditions for fundamental frequency, sub-harmonic of 2nd order and 3rd order forced oscillations of a second order feedback control system are investigated; also the fundamental frequency forced oscillation for a higher order system and the jump resonance frequencies of response curve are investigated, and a general expression for the equivalent gain of the nonlinearity has been developed.

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LIST OF SYMBOLS

e, E, $e(t)$, $E(t)$	Error signal
e_i	Input of nonlinearity
e_o	Output of nonlinearity
E_1	Amplitude of 1st harmonic error signal
$E_{1/2}$	Amplitude of 2nd sub-harmonic error signal
$E_{1/3}$	Amplitude of 3rd order sub-harmonic error signal
F	Amplitude of input function
$f(E)$	Restoring force function of nonlinearity
$G(s)$	Open loop transfer function in forward path
G_n, N	Nonlinear element
$H(s)$	Transfer function of feedback element
I	Imaginary part of open loop transfer function vector
K	System constant in forward path
M	Magnification factor
$N(E_1)$	Equivalent gain of nonlinearity
n	Integer
R	Real part of open loop transfer function vector
S_r	The root of characteristic equation of linear system
t	Time
ω	Frequency
ω_n	Natural frequency
$\theta_c, \theta_c(t)$	Output signal
θ_r, $\theta_r(t)$	Input force function
ζ	Damping factor
α	Attenuator factor
θ	Phase angle
$\bar{\theta}_c, \bar{\theta}_r, \bar{E}$	Transformed quantity

CHAPTER I

INTRODUCTION

One of the major branches of servo mechanism analysis for nonlinear control systems is the study and methods of prevention of continuous oscillation. For a linear control system, the output may oscillate with continuous or increasing amplitude with no input signal applied, in which case, the system is said to be unstable, or they may have oscillations which die away. In both cases this transient response is the characteristic of the system itself and the conditions for divergent, continuous or decaying oscillation do not depend on the form or magnitude of the input signal.

When a nonlinear element is present, this independence of the input signal no longer holds. The characteristic performance of the nonlinear control system will depend on both, i.e., forcing function and the characteristic of the nonlinear element.

A nonlinear control system, under suitable conditions may exhibit steady oscillations in which the main component has a frequency which is dependent on the frequency of the forcing function. This type of oscillation is called "Forced Oscillation". A forced oscillation for which the frequency is a fraction of the forcing function frequency is called "Subharmonic Forced Oscillation". A forced oscillation for which the frequency is multiple of the forcing frequency is called "Super Harmonic Forced Oscillation".

Since a nonlinear control system may possess more than one type of forced oscillations, it is highly desirable to investigate the type of forced oscillations which can occur when the nonlinear restoring force function is of a specific form.

Some methods of investigation for nonlinear forced oscillation have been discussed by J. C. West[1] in his book "Analytical Techniques for Nonlinear Control System" (1960). A graphic method for investigating forced oscillation of "ON OFF" control system has been developed by Hamel (French) and Tsypkin (USSR)[2]. Some work for subharmonic oscillations of a typical restoring force function has been done by C. A. Ludeke and William Pong[3] in their paper 1959, and also by Ogata in his PhD thesis[4], June 1956.

There are some other authors for investigation of the vibration of a mechanical system, such as N. Minorsky[5], J. J. Stoker[6], and Y. H. Ku[7] in their books of "Introduction to Nonlinear Mechanics";"Nonlinear Vibrations" and "Analysis and Control of Nonlinear System" respectively.

The purpose of this paper is to present a harmonic linearization method[8] for finding the existence of forced oscillations and response curve characteristics of a nonlinear feedback control system, by means of finding the restoring force function of the nonlinearity[9,10], and using an iteration method for investigating the condition for one type of forced oscillations exhibited. A response curve and "Jump Resonance" can be also investigated.

The emphasis of this paper is placed on showing the general approach of the method, therefore, second order nonlinear feedback control systems are used as examples. However, a higher order system will be discussed in this paper also.

By the harmonic linearization and iteration method, a general equation for the equivalent gain of an odd function nonlinear element will be developed[11]. It is in terms of the amplitude and the frequency of input.

There are six chapters in the main body of this paper. The first chapter and the last chapter are a general introduction and conclusion respectively. Chapter two is a general description of oscillation of feedback control systems and an investigation of the restoring force function of the nonlinear element. Chapter three investigates the existence of fundamental forced oscillations and the jump phenomena in the nonlinear feedback control system. Chapter four is a sub harmonic oscillation investigation. Chapter five investigates the existence of forced oscillation of a higher order feedback control system.

CHAPTER II

GENERAL DISCUSSION OF OSCILLATION IN FEEDBACK CONTROL SYSTEMS

2-1 General Description:

For either a linear or nonlinear feedback control system, most requirements for control system design specify a minimum time response, high accuracy, and high stability. It is clearly in order to have a minimum time response and high accuracy for a feedback control system, but first of all, the system should be stable.

In general, two types of oscillations may be exhibited in a feedback control system, one is free oscillation, i.e., an autonomous system. This is a system which is left by itself with its outside source of energy suddenly cut off or abruptly changed in amplitude. In such a system, free oscillations occur as the system tends to equilibrium after reaching a state of unequilibrium. The oscillation frequency of this type depends only on the characteristic of system itself. Ordinarily, a free oscillation is a damped oscillation; as the time increases the amplitude decreases and damps out in steady state. For an ideal case, if it is a system without damping, i.e., $\alpha = 0$, the oscillation will be continuous.

The other type of oscillation is a forced oscillation, i.e. a nonautonomous system. This is a system that is acted upon by an outside source of energy, a force function. The force function may be a constant; a periodic function or any other function of time.

The analysis and methods for preventing oscillations of a feedback control system are concerned directly or indirectly with the characteristic of system, i.e., the differential equations of the system. If the system is linear a frequency response of the system can be directly solved from the differential equation, furthermore, the principle of superposition can be applied too. On the other hand, if the system is nonlinear, either with a nonlinear damping or nonlinear restoring force function, then the basic tools for the analysis of a linear system are no longer valid.

2-2 Oscillation in Linear Systems:

There are two types of methods for analysis and design of linear feedback control systems; one is graphic method, the other is an analytical method. The best known of graphic methods consist of Bode diagram; Polar plot; Nichols diagram; and Root locus. If we know the transfer function of the system, any of the above methods can be used for solving the problem of stability. The analytical method is a mathematical analysis for solving the differential equation, and plot the curves of the solved equation.

First consider a simple linear feedback control system, as shown in Fig. 2-1. The forward element has a transfer function G(s), and a feedback

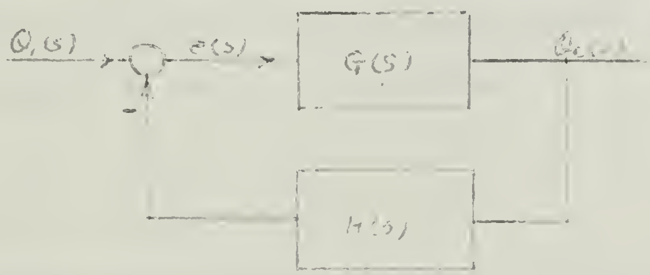


Fig. 2-1 Block Diagram of a Linear System

element transfer function H(s). The feedback signal is $H(s)\theta_c(s)$ and it is assumed that this is subtracted from the input signal $\theta_r(s)$ to form the control signal e(s); thus:

$$e(s) = \theta_r(s) - H(s)\,\theta_c(s) \qquad (2\text{-}1)$$

The relation between the control signal and reference signal in the Laplace transformation form is:

$$\theta_c(s) = \frac{G(s)}{1 + H(s)G(s)}\,\theta_r(s) \qquad (2\text{-}2)$$

The solution of equation (2-2) for a particular input can be obtained by Laplace and Heaviside methods. This involves the determination of the roots of equation:

$$1 + H(s)G(s) = 0 \tag{2-3}$$

In order to split the transfer function into the sum of partial fractions of standard form for which time solutions are known. If equation (2-3) is an n th order polynominal in s, there will be n roots say S_1 to S_n, such that $(s - S_r)$ is a factor of the polynominal. Special cases arise in which some of roots are identical, and if m roots of S_j occurs, $(s - S_j)^m$ is a factor of the polynominal.

In general, the roots S_r will be a complex number, as:

$$S_r = \alpha_r + j\,\omega_r \tag{2-4}$$

in which α_r and ω_r are both real numbers, and since the coefficients of the polynominal of equation (2-3) are all necessarily real, then the complex roots should be in conjugate pairs, Thus:

$$(s - \alpha_r + j\,\omega_r)\ (s - \alpha_r - j\,\omega_r)$$

will be a factor of the polynominal.

In determining the transient response to a forcing function input, these various factors become the denominator of the partial fraction representation of the transfer function equation (2-2). The term of $K_r/(s - S_r)$ gives a time solution containing expoential term:

$$e^{(\alpha_r + j\,\omega_r)t}$$

The complete solution is the sum of all such terms.

The complex conjugate pairs give rise to an oscillatory term:

$$e^{\alpha_r t}\ \ \mathrm{Sin}(\omega_r t + \phi)$$

If the system has convergent response, all of these terms in the time solution must converge and hence all roots S_1 to S_n must have negative real parts. If any of these roots has a zero real part, then a continuous oscillation is produced. If any of the complex roots has a positive real part, then divergent oscillation will be the result.

To determine the stability of a system by using the graphic method, it is not necessary to find the roots of the characteristic equation. If the equation is known algebrically, the Routh method can be applied and also by the Nyquist and Bode diagram method a graphical solution can be obtained.

Nyquist criteron graphical analysis is the most useful method to solve the stability problem of control systems. Rewriting equation (2-3) in the form of:

$$G(s)H(s) = -1 \tag{2-5}$$

The roots of equation of (2-3) are particular values of s which satisfy equation (2-5). Thus if the equation of G(s)H(s) can be mapped for continuously variable values of s, then those values for which $G(s)H(s) = -1$ can be determined.

Assume a particular roots of $S_r = \alpha_r + j\omega_r$ can be represented by a particular value of G(s)H(s), i.e.

$$G(\alpha_r + j\omega_r)H(\alpha_r + j\omega_r) = R(\alpha_r, \omega_r) + jI(\alpha_r, \omega_r) \tag{2-6}$$

This equation represents a vector in I vs R plane, the magnitude of vector:

$$M = \sqrt{R^2 + I^2} \tag{2-7}$$

and the direction by an angle:

$$\phi = \mathrm{Tan}^{-1} I/R \tag{2-8}$$

For a constant α_r, the value of M will be a locus in I-R plane as ω varies. This process can be repeated for different values of α_r to obtain a family of curves as shown in Fig. 2-2.

This steady state frequency response locus of the open loop system can be obtained experimentally from the system by measuring gain and phase for a sinusodial input of varying frequency. It can be shown that is the steady state frequency response locus passes to the right side of (-1, 0)

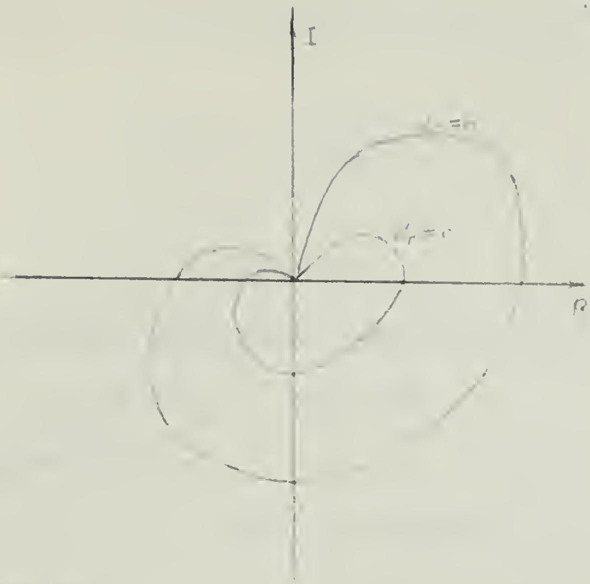


Fig. 2-2 Locus of an Open Loop Transfer Function G(s)H(s)

point, this system will be stable, on the other hand, if it passes through or to the left side of (-1, 0) point, the system will be unstable. This criterion is due to Nyquist. A typical curve is shown in Fig. 2-3.

In the extension to nonlinear control systems, or more complex multi-loop systems, the single loop criterion is sufficient for the majority of needs where G(s) can represent the over all transfer function of several loops provided that G(s) is itself stable.

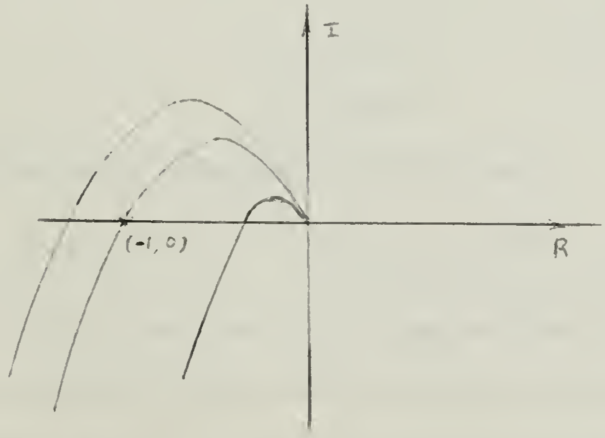


Fig. 2-3 Nyquist Criterion for Three Characteristic States

2-3 Forced Oscillations in a Linear System:

Assume a second order linear system block diagram is shown in Fig. 2-1; in which:

$$G(s) = \frac{K}{s(s + \alpha)} \tag{2-9}$$

and:

$$H(s) = 1 \tag{2-10}$$

From equation (2-2), the differential equation of system becomes:

$$\ddot{\theta}_c + \alpha \dot{\theta}_c + K\theta_c = K\theta_r \tag{2-11}$$

Let the forcing function:

$$\theta_r(t) = F \cos \omega t \tag{2-12}$$

Equation (2-11) becomes:

$$\ddot{\theta}_c + \alpha \dot{\theta}_c + K\theta_c = C \cos \omega t \tag{2-13}$$

in which C is a constant of value of FK.

The solution of equation (2-13) consists of the sum of the solution of the homogeneous equation, (i.e. the free oscillation of the system) and the solution of the non-homogeneous equation. Let the solution of homogeneous equation:

$$\theta_c(t) = e^{-\frac{\alpha}{2}t}(C_1 \cos \omega_n \sqrt{1-\rho^2}\, t + C_2 \sin \omega_n \sqrt{1-\rho^2}\, t) \tag{2-14}$$

in which:

$$\omega_n = \sqrt{K}$$

$$\alpha = 2\rho\omega_n$$

$$\rho = \cos \gamma$$

The complete solution of equation (2-13) will be:

$$\theta_c(t) = e^{-\frac{\alpha}{2}t}(C_1 \cos \omega_n \sqrt{1-\rho^2}\, t + C_2 \sin \omega_n \sqrt{1-\rho^2}\, t) + \frac{C \cos(\omega t + \theta)}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\rho^2\omega_n^2\omega^2}} \tag{2-15}$$

Equation (2-15) is obtained by a superposition of the free oscillation and the forced oscillation which varies from the action of external force:

The frequency of forced oscillation is the same as that of the external force, the magnitude of the forced oscillation is given by:

$$H = \frac{C}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\beta^2 \omega_n^2 \omega^2}} \tag{2-16}$$

The value of phase shift θ related to the external force is given by:

$$\cos\theta = \frac{\omega_n^2 - \omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\beta^2 \omega_n^2 \omega^2}} \tag{2-17}$$

$$\sin\theta = \frac{2\beta \omega_n \omega}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\beta^2 \omega_n^2 \omega^2}} \tag{2-18}$$

In the case of positive damping, i.e. $\alpha > 0$, it is clear from equation (2-15) that after a sufficiently long period of time, the free oscillation is damped out and only the forced oscillation would be observed.

In the case of no damping, i.e. $\alpha = 0$, the phase shift θ is seen from equation (2-17) and (2-18) to be zero for $\omega < \omega_n$ and π for $\omega > \omega_n$. In other words, the forced oscillation is in phase with the external force if the forced oscillation frequency is less than the free oscillation frequency, and is 180 degrees out of phase with the external force when ω is greater than ω_n.

In the case of $\alpha = 0$, equation (2-15) becomes:

$$\theta_c(t) = (C_1 \cos\omega_n t + C_2 \sin\omega_n t) + \frac{C \cos(\omega t + \theta)}{|\omega_n^2 - \omega^2|} \tag{2-19}$$

There will be a superposition oscillation of two frequencies, if the value of $\omega \neq \omega_n$ one of which is the natural frequency, and the other is the frequency of external force. In the case of $\omega = \omega_n$, the free and the forced oscillations have the same frequency, and the solution of equation (2-13) will be found:

$$\theta_c(t) = C_1 \cos\omega t + C_2 \sin\omega t - \frac{C}{2\omega} t \sin\omega t \tag{2-20}$$

In this case, the component due to the external force is no longer periodic, it is oscillatory with an amplitude that increases linearly with time. At this condition, it is a resonance phenomenon. It is very important in design to avoid this phenomenon in a system with a periodic force function.

In the case of a damped system, i.e., $\alpha > 0$, from equation (2-16) for the steady state, the amplitude of forced oscillation is always finite. It is possible to give, by the use of dimensionless variable, a more general significance to the expression for the amplitude of forced oscillation, let the amplitude:

$$|H| = M\left(\frac{C}{K}\right) \tag{2-21}$$

Therefore equation (2-16) becomes:

$$M = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + (2\rho)^2\left(\frac{\omega}{\omega_n}\right)^2}} \tag{2-22}$$

Where M is defined to be the magnification factor. The extreme values for M are attained for $\omega = 0$ and $(\omega / \omega_n)^2 = 1 - 2\rho^2$. If $1 - 2\rho^2 < 0$, there is a maximum for $\omega = 0$; if $1 - 2\rho^2 > 0$, and $\omega > 0$, there is a maximum for $\omega/\omega_n = \sqrt{1-2\rho^2}$ and a minimum for $\omega = 0$. For small values of damping coefficient, the frequency which produced the maximum amplitude is very nearly the natural frequency of the system. Fig. 2-4 shows the forced oscillation response curves of a linear system as a function of ω/ω_n with various values of ρ .

2-4 Oscillations in Nonlinear Feedback Control System:

As mentioned in section 2-1 of this chapter, when a nonlinearity is present in a feedback control system, the properties of proportionality and superposition are not valid. In the linear system, these properties involve the existence of a transfer function and of characteristic frequencies proper to the system (frequencies at which the system tends to oscillate with an amplitude depending on initial conditions); in the case of nonlinear systems, the amplitude, like frequency, can depend at one and the same time on both the initial conditions and the system itself. Sometimes, as in the case of limit cycle both frequency and amplitude of input are characteristic of the system and independent of of initial conditions.

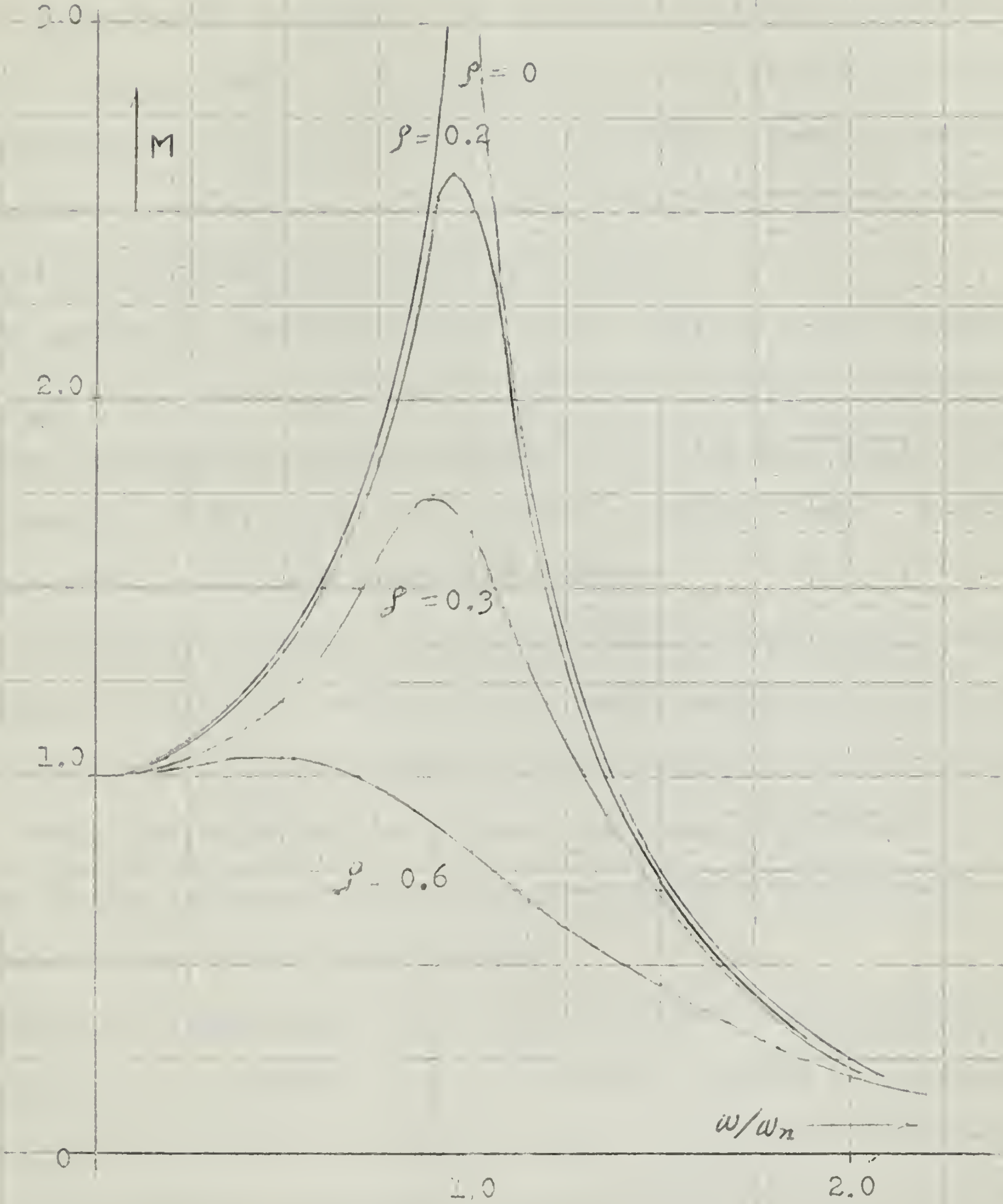


Fig.2-4 Response Curve For a Linear Forced Oscillations

In many cases the analysis and design of such systems using linear theory may produce an excellent result. This general procedure is a type of linearization procedure, i.e., the actual nonlinear system is replaced by a linear system, which approximates it, and the analysis and design techniques are applied to the linear equivalent. The most useful method for nonlinear linearization is that of the "Describing Function" method. It is actually a first harmonic approximation method.

The method for linearization used by this paper is a multi-harmonic linearization and iteration method; first assume a harmonic solution for the system differential equation and insert it in the differential equation, and compare the coefficients of the same order harmonic terms. A detailed procedure will be discussed in the next chapter.

As previously explained, a nonlinear feedback control system may posses some type of oscillation. Which type will exist depends on the restoring force function of nonlinear element and the amplitude of forcing function. Hence, before investigating the existence of forced oscillations, the restoring forcing function of the nonlinearity should be investigated.

2-5 Characteristic of Nonlinear Elements:

Nonlinear elements which may be seen in feedback control systems can be represented by a combination of four fundamental concepts of nonlinearity, which are: (1) Saturation; (2) Variable gain; (3) Dead zone; (4) Hysteresis. These basic notions of nonlinearity can, in combination with one another, result in almost all types of nonlinearity in practice, as shown in Fig. 2-5. For example an ideal relay nonlinearity can be seen as a saturation nonlinearity which linear part with a value of infinite slope.

The operational characteristic of a nonlinear element which operates in a control system depends on the operating condition, just as a vacuum tube for which the operating characteristic depends on the operating point of its

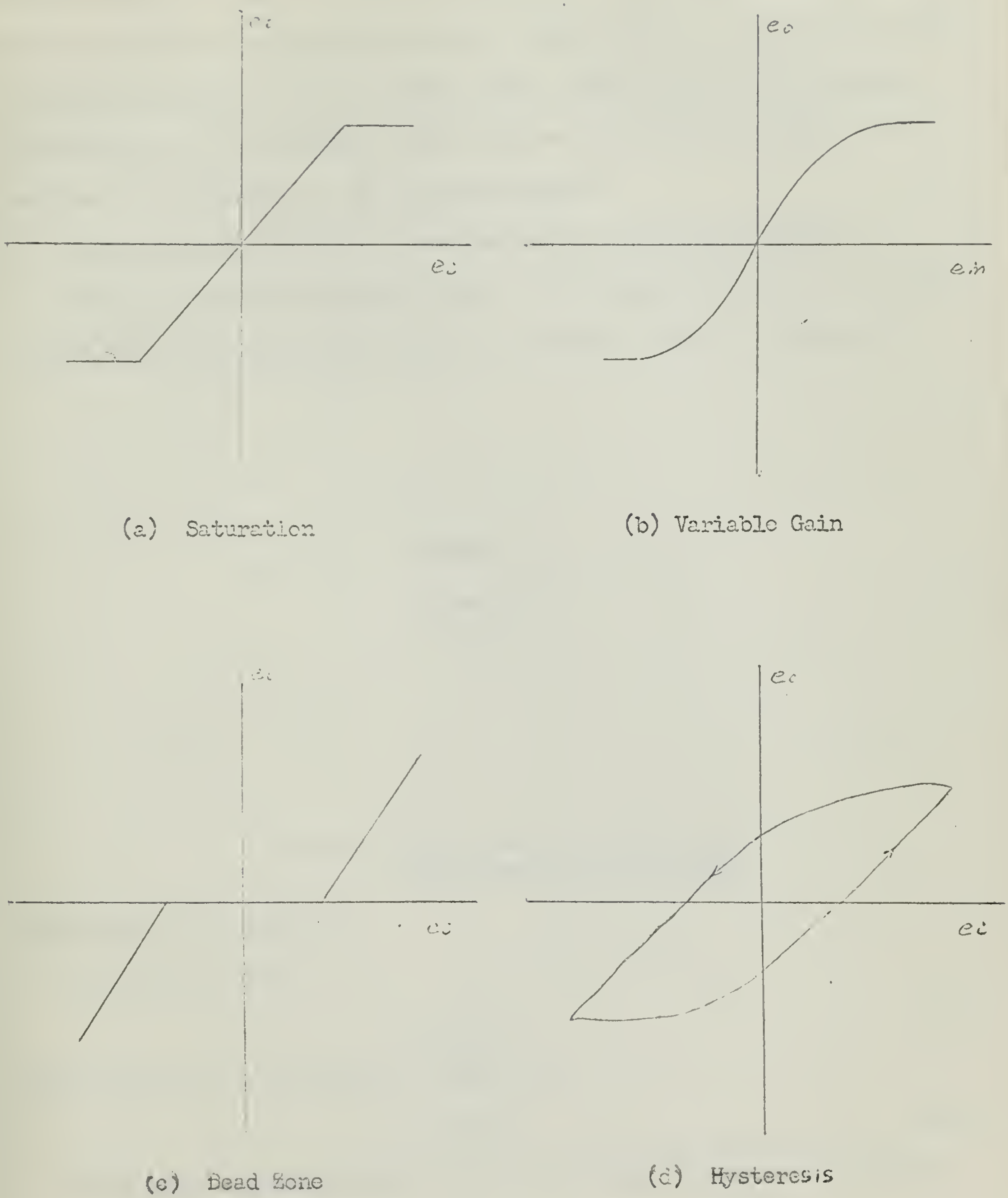


Fig. 2 5 Basic Concepts of Nonlinear Element characteristic curves. As a saturation nonlinearity element with a small forcing function input, it operates linearly. On the other hand, if the forcing function is very large and with a high frequency, the actual operating characteristic is like an ideal relay. Hence, the actual operating characteristic of a nonlinear element may be said to be a function of the amplitude and frequency of the forcing function.

2-6 Restoring Force Function Investigation of a Nonlinear Element:

Consider the block diagram of Fig. 2-6, in which the output is a function of input, usually this function in a control system is called the

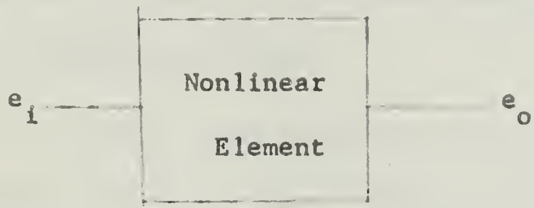


Fig. 2-6 Basic Relation Between Input and Output of a Nonlinearty

"Restoring Force Function".

Assume the input is:

$$e_i = E(t) \tag{2-23}$$

The output will be a function of input, as:

$$e_o = f(E) \tag{2-24}$$

From the numerical analysis and curve fitting[9] process, the function of f(E) can be expressed:

$$\begin{aligned} f(E) = a_o + a_1E + a_2E^2 + a_3E^3 + \dots\dots\dots\dots + a_nE^n \\ + b_1E + b_2E^{1/2} + b_3E^{1/2} + \dots\dots\dots\dots + b_nE^{1/n} \end{aligned} \tag{2-25}$$

Equation (2-25) can be in the general form:

$$f(E) = a_o + \sum_{i=1}^{n} a_i E^i + \sum_{i=1}^{n} b_i E^{1/i} \qquad (2\text{-}25a)$$

for this paper, the nonlinear characteristic is considered symmetrical to the original point, that means:

$$f(-E) = -f(E) \qquad (2\text{-}26)$$

It is an odd function, the coefficients of even order of equation (2-25) are zero, it becomes:

$$f(E) = a_o + a_1E + a_3E^3 + a_5E^5 + \ldots\ldots\ldots\ldots\ldots$$
$$+ b_1E + b_3E^{1/3} + b_5E^{1/5} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2\text{-}27)$$

For the saturation and variable gain nonlinearity, J. C. West[1] uses the restoring force function as a form:

$$f(E) = a_1E + a_3E^3 \qquad (2\text{-}28)$$

in which a_1 is greater or equal to zero, and a_3 is either greater, or less, or equal to zero corresponding to the case of a "Hard Spring", "Soft Spring", and "Linear Spring" saturation nonlinearity characteristic respectively. A set curves for all values of a_3 is shown in Fig. 2-7.

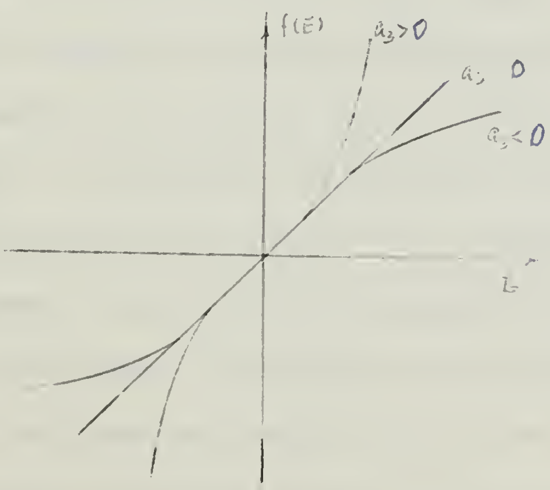


Fig. 2-7 Characteristic curves of Eq. (2-28)

In order to generalize for the condition of saturation nonlinearity with dead zone, equation (2-28) can be written with a constant as:

$$f(E) = -a_o \text{Sign } E + a_1 E + a_3 E^3 \qquad (2\text{-}29)$$

if the dead zone is less than one, the characteristic curve can be shown in Fig. 2-8.

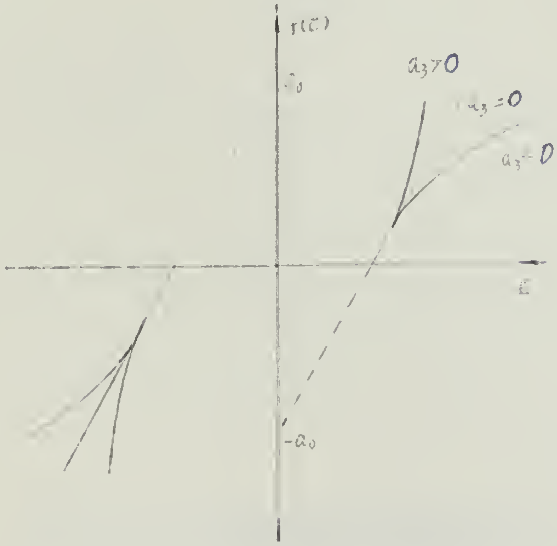


Fig. 2-8 Characteristic Curves of Eq. (2-29).

As discussed previously, when a saturation nonlinearity is operating with a larger amplitude than the saturation voltage and a high frequency, the operating characteristic is like an ideal relay. On the other hand, under some suitable conditions a relay nonlinearity control system may exhibit some sub-harmonic forced oscillations, as proved by the paper of A. M. Hopkin and K. Ogata[12]. This result is the same as a saturation nonlinearity.

Actually, the operation of a relay is not a perfect discontinuous characteristic element, when the relay is operating from static to closed, there will be a little amount of time delay from starting to close contacts. Therefore, the operating characteristic of an ideal relay may be considered a saturation nonlinearity with a very large slope of the linear part, a sketch of it is shown in Fig. 2-9.

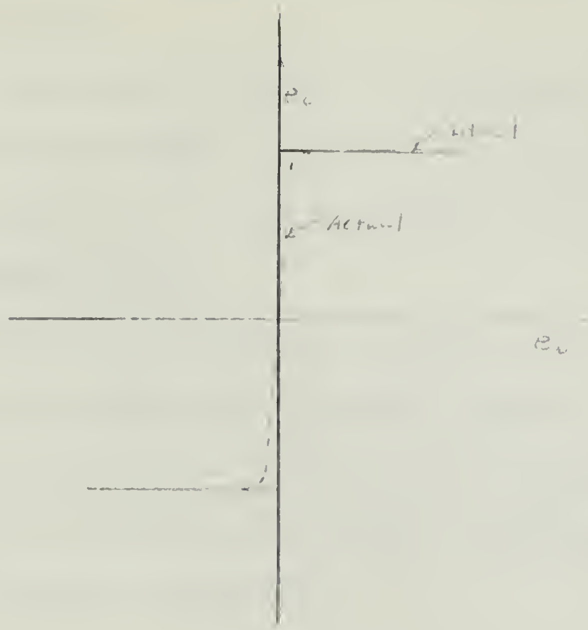


Fig. 2-9 Operating Characteristic of an Ideal Relay

From the above discussion, and for a proper value of constants, the restoring forcing function can be expressed as:

$$f(E) = a_1 E + a_3 E^3 + a_5 E^5 \tag{2-30}$$

Similarly, if we consider the dead zone the equation of (2-30) can be expressed:

$$f(E) = -a_o \text{ Sign}E + a_1 E + a_3 E^3 + a_5 E^5 \tag{2-31}$$

M. J. Abzug[13] investigates the restoring forcing function of an ideal relay nonlinearity by a fifth root or cube root of the input as:

$$f(E) = b_5 E^{1/5} \tag{2-32}$$

or;

$$f(E) = b_3 E^{1/3} \tag{2-32}$$

A more detailed discussion about frequency response by using the restoring forcing function will be covered in the next two chapters.

CHAPTER III

FUNDAMENTAL FREQUENCY FORCED OSCILLATION IN NONLINEAR CONTROL SYSTEMS

3-1 General Description:

Consider a system shown in Fig. 3-1, in which $G_1(s)$ is the transfer function of a controller element. $G_2(s)$ is the transfer function of a motor and gear element, G_n is a nonlinear element. H(s) denotes a transfer function of a feedback network.

When a periodic forcing function is applied to the input, the output of the system may be or may not be a periodic function. If it is a periodic output and the frequency is the same as the frequency of the input forcing function, it is said that the system is operating in forced oscillation with the fundamental frequency.

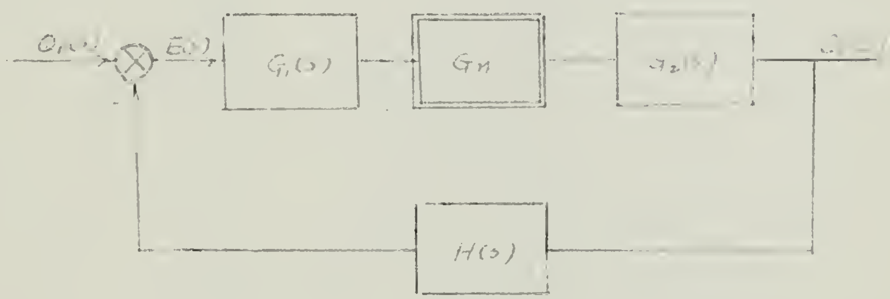


Fig. 3-1 Block Diagram of a Feedback Control System

There are some methods to represent the forced oscillation in a nonlinear system. If we keep an input forcing function with constant frequency and vary the amplitude of input, a response curve can be shown in the "Input vs Error" plane, Fig. 3-2, in which there is a cut off amplitude; that means when the input is smaller than θ_{r-m}, there is no forced oscillation.

The other representation is with the input amplitude constant vary

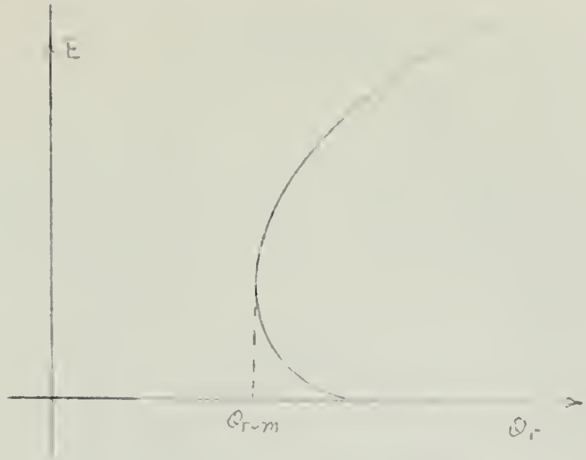


Fig. 3-2 Forced Oscillation Represented in the E vs θ_r Plane

the frequency of input. It is shown in "Error vs Frequency" plane. Fig. 3-2a is an example, it is actually a closed loop frequency response of the system.

Fig. 3-2a Forced Oscillation Represented in the E vs ω Plane

Either represented in E vs θ_r or E vs ω plane, both of them can be represented in a phase plane ($\dot{E}$ vs E). A typical phase plane of forced oscillation from the computer is shown in Fig. 3-3, in which it is represented by an ellipse. The size of it is changed with the frequency.

3-2 Basic Equation for Forced Oscillation of 2nd Order Nonlinear Feedback Control System:

Consider a 2nd order feedback control system shown in Fig. 3-4:

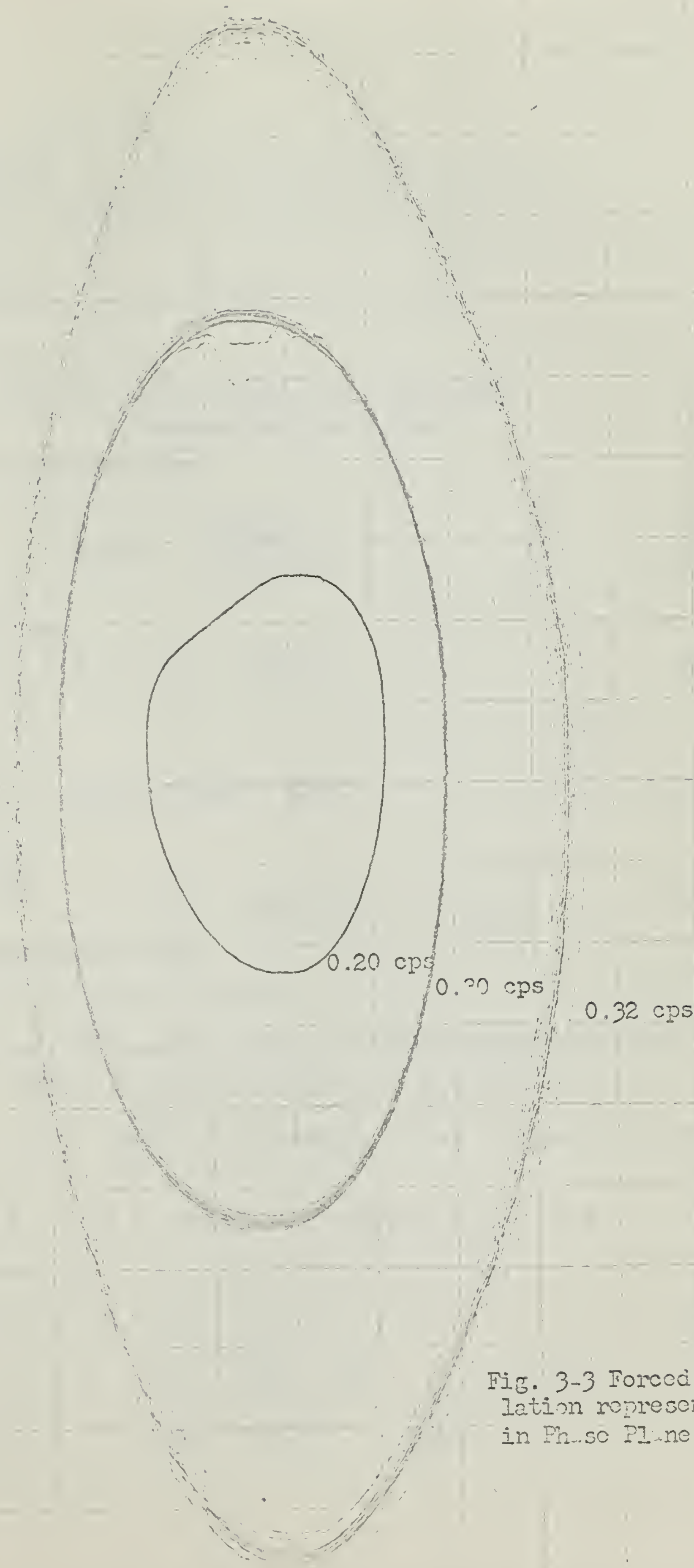


Fig. 3-3 Forced Oscillation represented in Phase Plane.

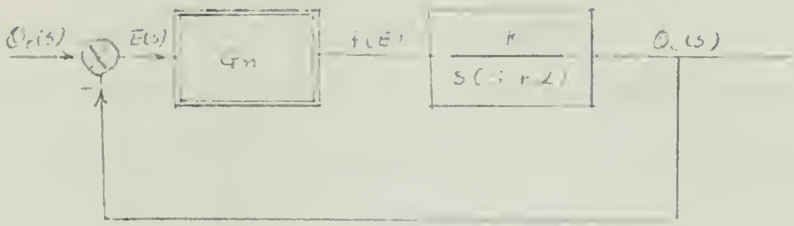


Fig. 3-4 Block Diagram of a 2nd Order Feedback Control System

The output equation in Laplace form:

$$\theta_c(s) = \frac{Kf(E)}{s(S+\alpha)} \tag{3-1}$$

in the differential form:

$$\ddot{\theta}_c + \alpha\,\dot{\theta}_c = Kf(E) \tag{3-2}$$

Where:

$$\theta_c(s) = \theta_r(s) - E(s) \tag{3-3}$$

Equation (3-2) becomes:

$$\ddot{E} + \alpha\,\dot{E} + Kf(E) = \ddot{\theta}_r + \alpha\,\dot{\theta}_r \tag{3-4}$$

Assume the input forcing function:

$$\theta_r(t) = F\,\mathrm{Cos}(\omega t + \theta) \tag{3-5}$$

in which θ is the phase difference between the error signal and forcing function of input. Inserting equation (3-5) in to equation (3-4):

$$\ddot{E} + \alpha\,\dot{E} + Kf(E) = -A\,\mathrm{Cos}(\omega t + \theta) - B\,\mathrm{Sin}(\omega t + \theta) \tag{3-6}$$

or:

$$\ddot{E} + \alpha\,\dot{E} + Kf(E) = C\,\mathrm{Cos}(\omega t + \pi + \theta + \emptyset) \tag{3-6a}$$

Where:

$$A = \omega^2 F \tag{3-7}$$

$$B = \alpha\,\omega\,F \tag{3-8}$$

$$C = \sqrt{A^2 + B^2} = F\omega\sqrt{\alpha^2 + \omega^2} \tag{3-9}$$

$$\emptyset = \mathrm{Tan}^{-1} B/A = \mathrm{Tan}^{-1} \alpha/\omega \tag{3-10}$$

Equation (3-6) or (3-6a) is a basic differential equation of a 2nd order nonlinear feedback control system, in which f(E) is the nonlinear restoring force function, and K is a constant of system.

3-3 Forced Oscillation; Saturation Nonlinearity Without Damping:

Consider a system without damping, i.e. $\alpha = 0$, equation (3-6) becomes:

$$\ddot{E} + Kf(E) = -A \operatorname{Cos}(\omega t + \theta) \tag{3-11}$$

Let the restoring force function:

$$f(E) = a_1 E + a_3 E^3 \tag{2-28}$$

Equation (3-11):

$$\ddot{E} + K(a_1 E + a_3 E^3) = -A \operatorname{Cos}(\omega t + \theta) \tag{3-12}$$

By the harmonic iteration method [7,8], the solution of equation (3-12) can be expressed as a series of odd order periodic harmonics:

$$e(t) = E_1 \operatorname{Cos} \omega t + E_3 \operatorname{Cos} 3\omega t + E_5 \operatorname{Cost} \omega t \ldots\ldots \tag{3-13}$$

in which E_1, E_3, E_5 ....are the amplitude of harmonics of e(t).

If we consider only the fundamental frequency forced oscillation, we can assume one solution is:

$$e(t) = E_1 \operatorname{Cos} \omega t \tag{3-14}$$

Where E_1 is the fundamental frequency amplitude of error signal, ω is the frequency of forcing function and forced oscillation; is to be determined. Since:

$$E = -\omega^2 E_1 \operatorname{Cos} \omega t \tag{3-15}$$

$$E = -\omega E_1 \operatorname{Sin} \omega t \tag{3-16}$$

$$E^3 = \frac{1}{4} E_1^3 (3 \operatorname{Cos} \omega t + \operatorname{Cos} 3\omega t) \tag{3-17}$$

Inserting equation (3-14) into (3-17) in equation (3-12):

$$(-\omega^2 E_1 + Ka_1 E_1 + \frac{3}{4} Ka_3 E_1^3) \operatorname{Cos} \omega t + \frac{1}{4} a_3 E_1^3 \operatorname{Cos} 3\omega t \tag{3-18}$$

$$= -A (\operatorname{Cos} \omega t \operatorname{Cos}\theta + \operatorname{Sin} \omega t \quad \operatorname{Sin}\theta)$$

neglecting the higher order harmonic terms and equating the coefficients of $\operatorname{Cos} \omega t$ and $\operatorname{Sin} \omega t$, thus:

$$\left(-\omega^2 E_1 + K a_1 E_1 + \frac{3}{4} K a_3 E_1^3\right) = -A\cos\theta \quad (3\text{-}19)$$

$$A\sin\omega t \ \sin\theta = 0 \quad (3\text{-}20)$$

Equation (3-20), the term of $A\sin\omega t$ is not always equal to zero, the term of $\sin\theta$ should be zero. Hence the value of θ will be either zero or π. Inserting equation (3-7) into equation (3-19):

$$\omega = \left(\frac{K a_1 E_1 + \frac{3}{4} K a_3 E_1^3}{E_1 - F}\right)^{1/2} \quad (3\text{-}21)$$

For $\theta = 0$, and:

$$\omega = \left(\frac{K a_1 E_1 + \frac{3}{4} K a_3 E_1^3}{E_1 + F}\right)^{1/2} \quad (3\text{-}22)$$

for $\theta = \pi$.

As a check, in the case of $F = 0$, i.e. no forcing function input, equations (3-21) and (3-22) give the free oscillation frequency of system:

$$\omega = \sqrt{K}\left(a_1 + \frac{3}{4} a_3 E^2\right)^{1/2} \quad (3\text{-}23)$$

For the linear case, i.e., $a_3 = 0$, the natural frequency :

$$\omega_n = (a_1 K)^{1/2} \quad (3\text{-}24)$$

It is exactly the same as we discussed in the linear system.

Recall equations (3-21) and (3-22); the value of ω should be real, that means:

$$a_1 + \frac{3}{4} a_3 E_1^2 > 0 \quad (3\text{-}25)$$

for both the phase angle is either zero or π. And :

$$E_1 > F \quad (3\text{-}26)$$

for the phase angle is zero only.

Equations (3-25) and (3-26) are the conditions for existence of the fundamental frequency forced oscillation.

A free oscillation frequency response, i.e. $F = 0$ can be sketched from equation (3-23), any value of force function other than zero also can be sketched on both sides of the response curves of $F = 0$. It is to be noted, that the phase angle between the force function and the error signal is opposite, i.e., $\theta = \pi$, when the response curves are to the left of curve for $F = 0$, and the phase angle between them is in phase, when the response curves are to the right of it. In other words, it is according to whether the frequency is less or greater than the frequency of free oscillation for that particular amplitude of error signal with a constant of force function input.

A typical error vs frequency ($E_1 \sim \omega$) response curves and phase relation sketch for a different value of forcing function and different characteristic of nonlinearity are indicated schematically in Fig. 3-5. The response curve for free oscillation, corresponding $F=0$, is drawn as a dash line.

In this respect the behavior of the nonlinear oscillation is the same as that of the linear oscillation. One sees that the response curve in the nonlinear cases could be thought of as arising from those for the linear case by bending the latter to the right for a hard spring saturation nonlinearity, and to the left for a soft spring saturation nonlinearity.

A response curve in "Input vs Frequency" plane ($\theta_c \sim \omega$) can be also obtained from relation of:

$$\theta_c(t) = \theta_r(t) - e(t) \tag{3-27}$$

or:

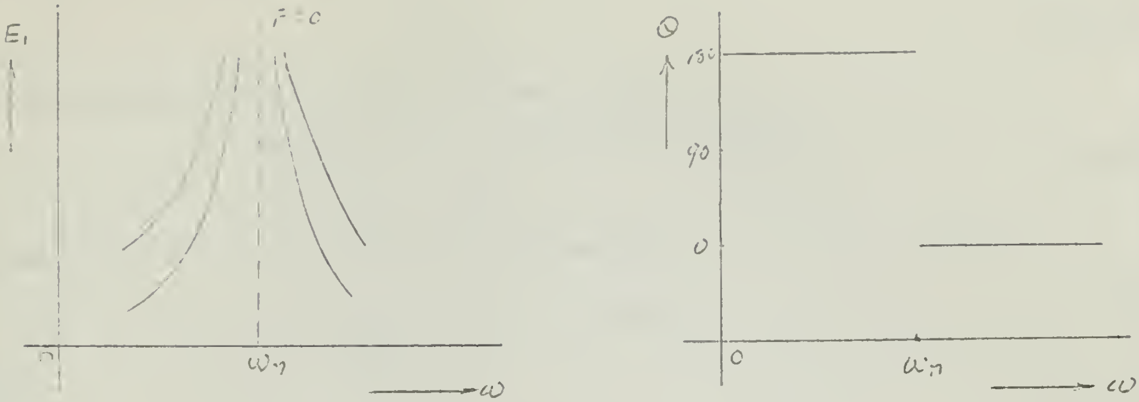
$$\theta_c(t) = F \cos(\omega t + \theta) - E_1 \cos \omega t \tag{3-28}$$

In the case of $\theta = 0$, equation (3-28) becomes:

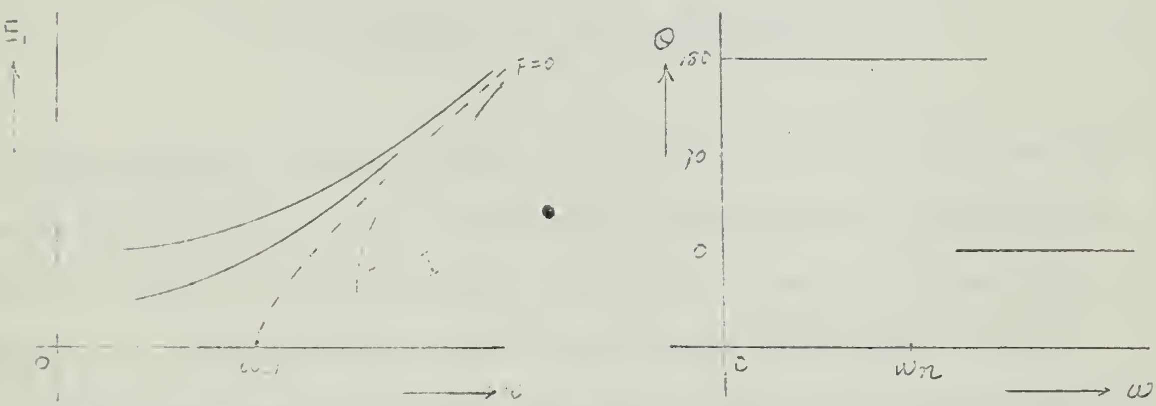
$$\theta_c(t) = (F - E_1) \cos \omega t \tag{3-29}$$

Hence:

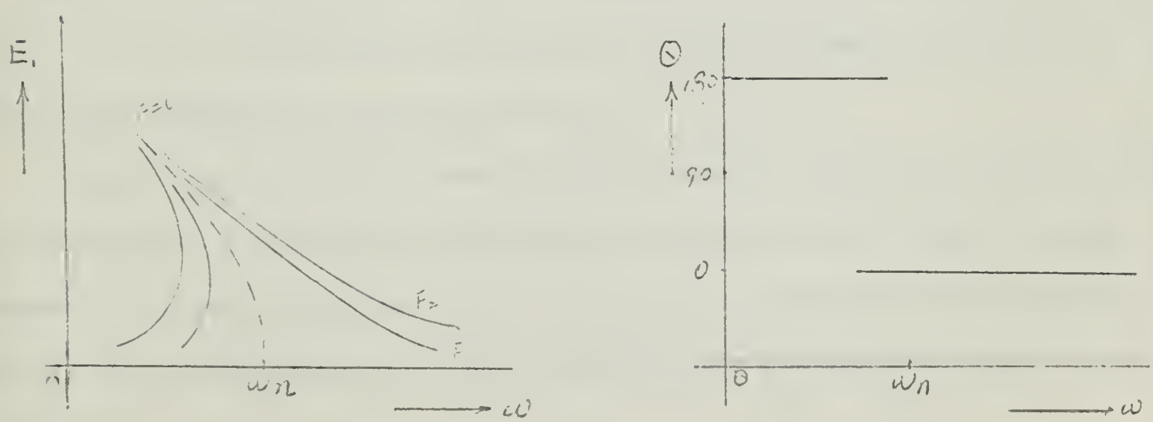
$$\theta_c = F - E_1 \tag{3-30}$$



(a) Linear System, $a_3 = 0$



(b) Hard Spring Saturation, $a_3 > 0$



(c) Soft spring Saturation, $a_3 < 0$

Fig. 35 Response Curves for the Saturation Nonlinearity Without Damping or:

$$E_1 = F - \theta_c \tag{3-31}$$

Inserting equation (3-31) into equation (3-21):

$$\omega = \left(\frac{Ka_1 (\theta_c - F) + \frac{3}{4} Ka_3 (\theta_c - F)^3}{\theta_c} \right)^{1/2} \tag{3-32}$$

For the case of $\theta = \pi$, equation (3-28) becomes:

$$\theta_c(t) = (F + E_1) \operatorname{Cos} \omega t \tag{3-33}$$

Hence:

$$E_1 = - (F + \theta_c) \tag{3-34}$$

Inserting equation (3-34) into equation (3-23):

$$\omega = \frac{K \quad a_1(\theta_c + F) + \frac{3}{4} \; a_3 \; (\theta_c + F)^3}{\theta_c} \; 1/2 \tag{3-35}$$

From equations (3-32 and (3-35), if $F = 0$, the result is the same form as equation (3-23), that means the response curve for $F = 0$ is the same as plotted in the $E_1 \sim \omega$ plane, only the difference is that the value of E_1 changes to θ_c. The response curves for the values other than $F = 0$ are changed, and the phase relationship between the input force function and the output is reversed.

There is a numerical example for plotting the response curve in $E_1 \sim \omega$ and $\theta_c \sim \omega$ plane respectively in section 3-7 of this chapter, and also experimental response curves are plotted.

From the analysis of the above and the numerical example in section 3-7, it is shown that a saturation nonlinearity feedback control system can have fundamental frequency oscillations. This not only depends on the characteristic of the nonlinearity, but also depends on the amplitude of input force function.

3-4 Forced Oscillations: Saturation Nonlinearity with Damping.

As explained previously, if there is no damping present in the system, there is an oscillation error signal $e(t) = E_1 \operatorname{Cos}(\omega t)$ either in phase or out of phase of 180 degrees with the forcing function of input. In the case where damping is present however, the forced oscillation displacement and the forcing input can be expected to be out of phase, just as in the corresponding of linear system.

Recall the basic equation (3-6) and the restoring force function equation (2-28):

$$\ddot{E} + \alpha\dot{E} + Kf(E) = -A\,Cos(\omega t + \theta) - BSin(\omega t + \theta) \qquad (3\text{-}6)$$

$$f(E) = a_1E + a_3E^3 \qquad (2\text{-}28)$$

Then equation (3-6) becomes:

$$\ddot{E} + \alpha\dot{E} + Ka_1E + Ka_3E^3 = -ACos(\omega t + \theta) - BSin(\omega t - \theta) \qquad (3\text{-}36)$$

Recall one solution of equation (3-36) is:

$$e(t) = E_1\,Cos\,\omega t \qquad (3\text{-}15)$$

Inserting equation (3-15) into equation (3-36):

$$(-\omega^2E_1 + Ka_1E_1 + \frac{3}{4}Ka_3E_1^3)Cos\,\omega t - \alpha\omega E_1Sin\,\omega t + \frac{1}{4}Ka_3E_1^3Cos\,3\omega t$$

$$= (-ACos\theta - BSin\theta)\,Cos\,\omega t + (ASin\theta - B\,Cos\theta)\,Sin\,\omega t \qquad (3\text{-}37)$$

Neglect the higher order terms of harmonic and equates the coefficient of $Sin\,\omega t$ and $Cos\,\omega t$ respectively, hence:

$$(-\omega^2E_1 + Ka_1E_1 + \frac{3}{4}Ka_3E_1^3) = ACos\theta - BSin\theta \qquad (3\text{-}38)$$

$$-\alpha\omega E_1 = A\,Sin\theta - B\,Cos\theta \qquad (3\text{-}39)$$

Squaring equations (3-38) and (3-39), and adding:

$$(-\omega^2E_1 + Ka_1E_1 + \frac{3}{4}Ka_3E_1^3)^2 + \alpha^2\omega^2E_1^2 = A^2 + B^2 \qquad (3\text{-}40)$$

or:

$$(-\omega^2E_1 + Ka_1E_1 + \frac{3}{4}Ka_3E_1^3)^2 + \alpha^2\omega^2E_1^2 = \omega^4F^2 + \alpha^2\omega^2F^2 \qquad (3\text{-}41)$$

also:

$$\omega^4(E_1^2 - F^2) - \omega^2\left[2E_1^2(Ka_1 + \frac{3}{4}Ka_3E_1^2) - \alpha^2(E_1^2 - F^2)\right]$$

$$+ E_1^2(Ka_1 + \frac{3}{4}Ka_3E_1^2)^2 = 0 \qquad (3\text{-}42)$$

As a check of equations (3-38), (3-39) and (3-41); first consider the response curve in the high frequency and the low frequency ranges, the phase relation between the error signal or control signal and the input forcing function is either zero or 180 degrees. In this case, the sine terms of equation (3-38) and (3-39) are zero, the response curve is the same as equation (3-20). That means for a damped system if the phase angle between the input and the output is either zero or 180 degrees, the frequency response is exactly the same as the system with no damping.

On the other hand, if the value of damping is very small, the response curve will be very close to the sketch of those of Fig. 3-5. The only difference when very small damping is present is that the response curves are rounded off in the vicinity of curve for F = 0. A sketch of them is shown in Fig. 3-6.

For the phase investigation; recall equation (3-39) and rewrite in this form:

$$\sin\theta - \alpha\cos\theta = \alpha E_1/F \tag{3-43}$$

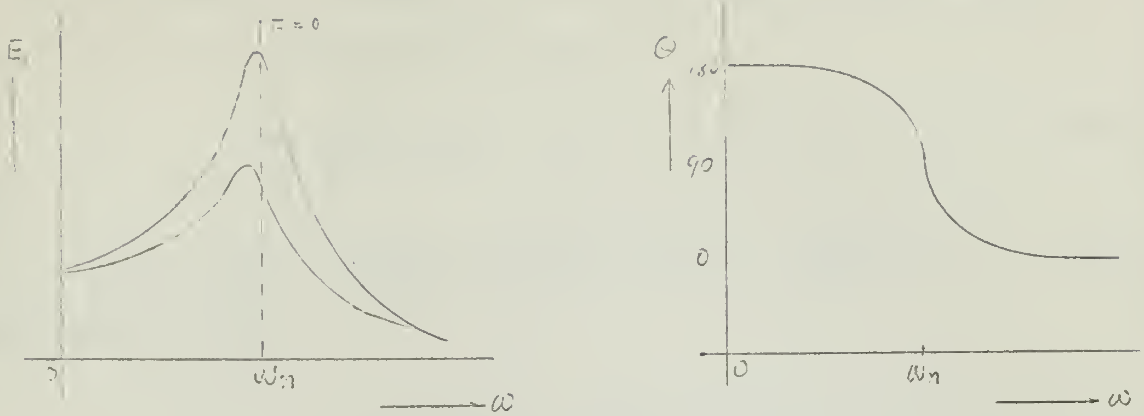
or:

$$\sin\left(\theta - \tan^{-1}\frac{\alpha}{\omega}\right) = \frac{\alpha E_1}{F(\omega^2 + \alpha^2)^{1/2}} \tag{3-44}$$

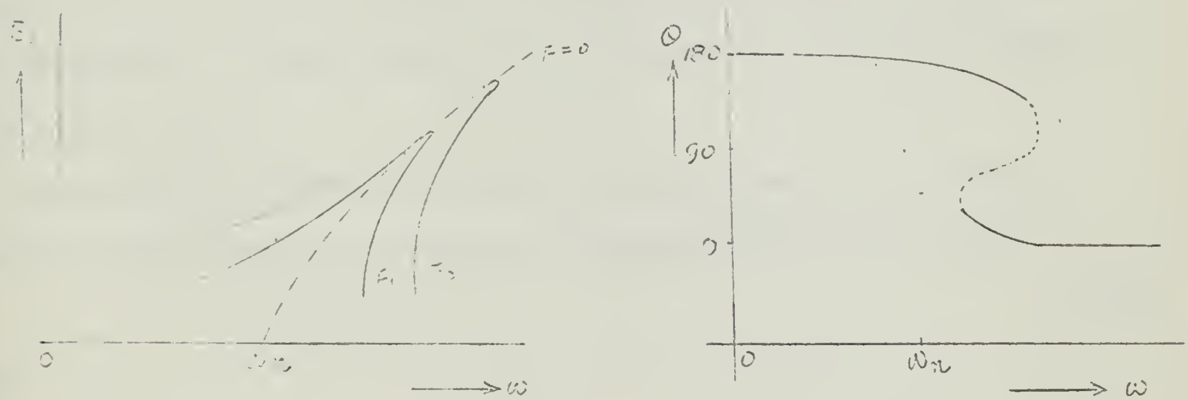
Hence, the phase angle between the error signal and the forcing function of input:

$$\theta = \sin^{-1}\frac{\alpha E_1}{F\,(\omega^2 + \alpha^2)^{1/2}} + \tan^{-1}\frac{\alpha}{\omega} \tag{3-45}$$

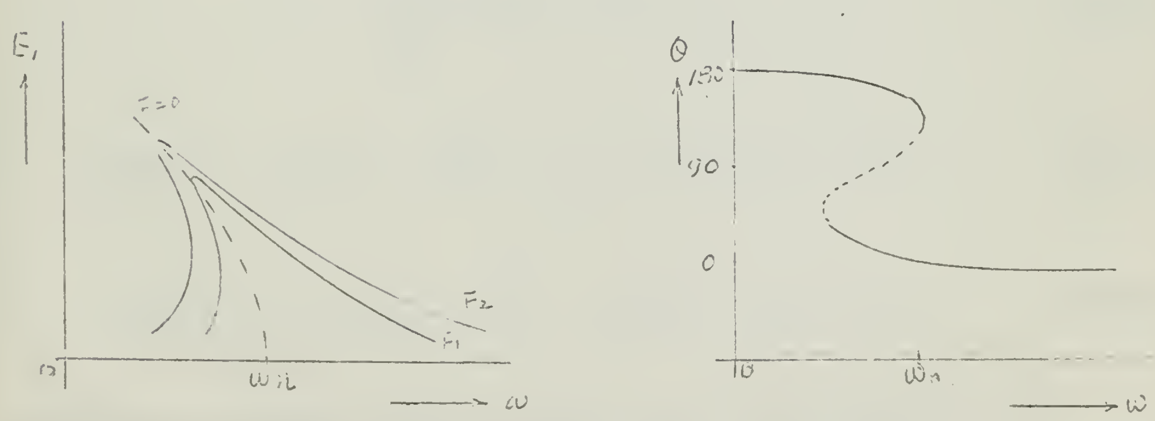
Examine equation (3-45) in the case of $\alpha = 0$, θ is either zero or π, when damping is present the value of θ is a function of ω and E_1, if the values of α and F are fixed. It should be noted, there are two values of phase shift for one value of error signal; one is for the low frequency ($\omega < \omega_n$) and the other is for the high frequency ($\omega > \omega_n$); one is from equation (3-45) and the other is from the 180 degrees minus the value of equation (3-45).



(a) Linear System, $a_3 = 0$.



(b) Hard Spring Saturation, $a_3 > 0$.



(c) Soft Spring Saturation, $a_3 < 0$.

Fig. 3-6 Response curves for The Saturation Nonlinearity with Damping

For the existence conditions for forced oscillation when damping is present, recall the equation (3-42), and put:

$$Ka_1 + \frac{3}{4} Ka_3E_1^2 = X \qquad (3\text{-}46)$$

and:

$$(E_1^2 - F^2)\omega^4 - [2E_1^2X - \alpha^2(E_1^2 - F^2)]\omega^2 + E_1^2X^2 = 0 \qquad (3\text{-}47)$$

Solve equation (3-47):

$$\omega = \sqrt{\frac{1}{2(E_1^2 - F^2)}\left\{[2E_1^2X - \alpha^2(E_1^2 - F^2)] \pm \sqrt{[(2E_1^2X - \alpha^2(E_1^2 - F^2)]^2 - 4E_1^2X^2(E_1^2 - F^2)}\right\}} \qquad (3\text{-}48)$$

It is to be noted that the conditions for forced oscillations in an undamped system are still valid for a damped system within the low and high frequencies, in other words, it is valid for the condition of phase angle either zero or π.

From equation (3-48) where forced oscillation exists, the value of should be real, hence; the condition for equation (3-48) to be real:

$$E_1^2 - F^2 > 0 \qquad (3\text{-}49)$$

and;

$$2E_1^2X - \alpha^2(E_1^2 - F^2) > 0 \qquad (3\text{-}50$$

or:

$$X > \frac{X}{2E_1^2}\alpha(E_1^2 - F^2) > 0 \qquad (3\text{-}51)$$

and;

$$[2E_1^2X - \alpha^2(E_1^2 - F^2)]^2 - 4E_1^2X^2(E_1^2 - F^2) > 0 \qquad (3\text{-}52)$$

or:

$$4E_1^2X^2F^2 + \alpha^4(E_1^2 - F^2) > 4E_1^2X^2\alpha^2(E_1^2 - F^2) \qquad (3\text{-}53)$$

Equation of (3-49); (3-51) and (3-53) are the conditions for the forced oscillation of a damped system.

3-5 Jump Resonance:

Recall a sketch of frequency response curve of a soft saturation nonlinearity shown in Fig. 3-7. When a experimental frequency response is being measured for a system, this measurement can be done by an analog computer to measure the amplitude of the input and the output, at the same time measure the phase relation between them.

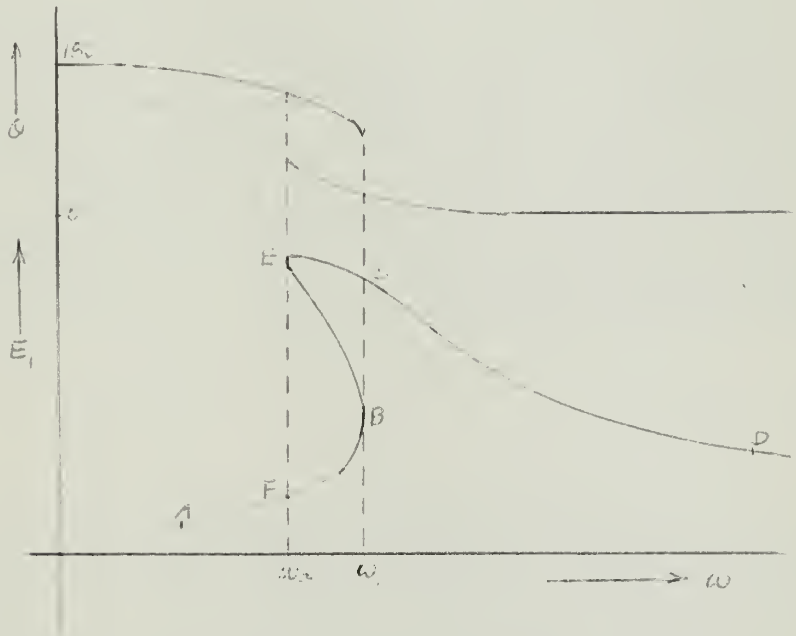


Fig. 3-7 The Region of Jump Resonance

It is found that under certain conditions of amplitude and frequency an infinitesmal change of either frequency or amplitude of the input signal causes a large and discontinuous jump in amplitude of the output, at the same time, a discontinuous phase jump occurs. Fig. 3-8 shows a set of amplitude jump and phase jump of a typical system from an Electronic Analog Computer experimental work.

Fig. 3-7 is shown as a typical system with a soft saturation nonlinearity and with a fixed amplitude of the input, which is larger than the value of saturation of nonlinearity. If we are going to measure the frequency response, being with a low frequency, then increase the frequency, given a response from A to B. At point B with a frequency ω_1 , an infinitesimal

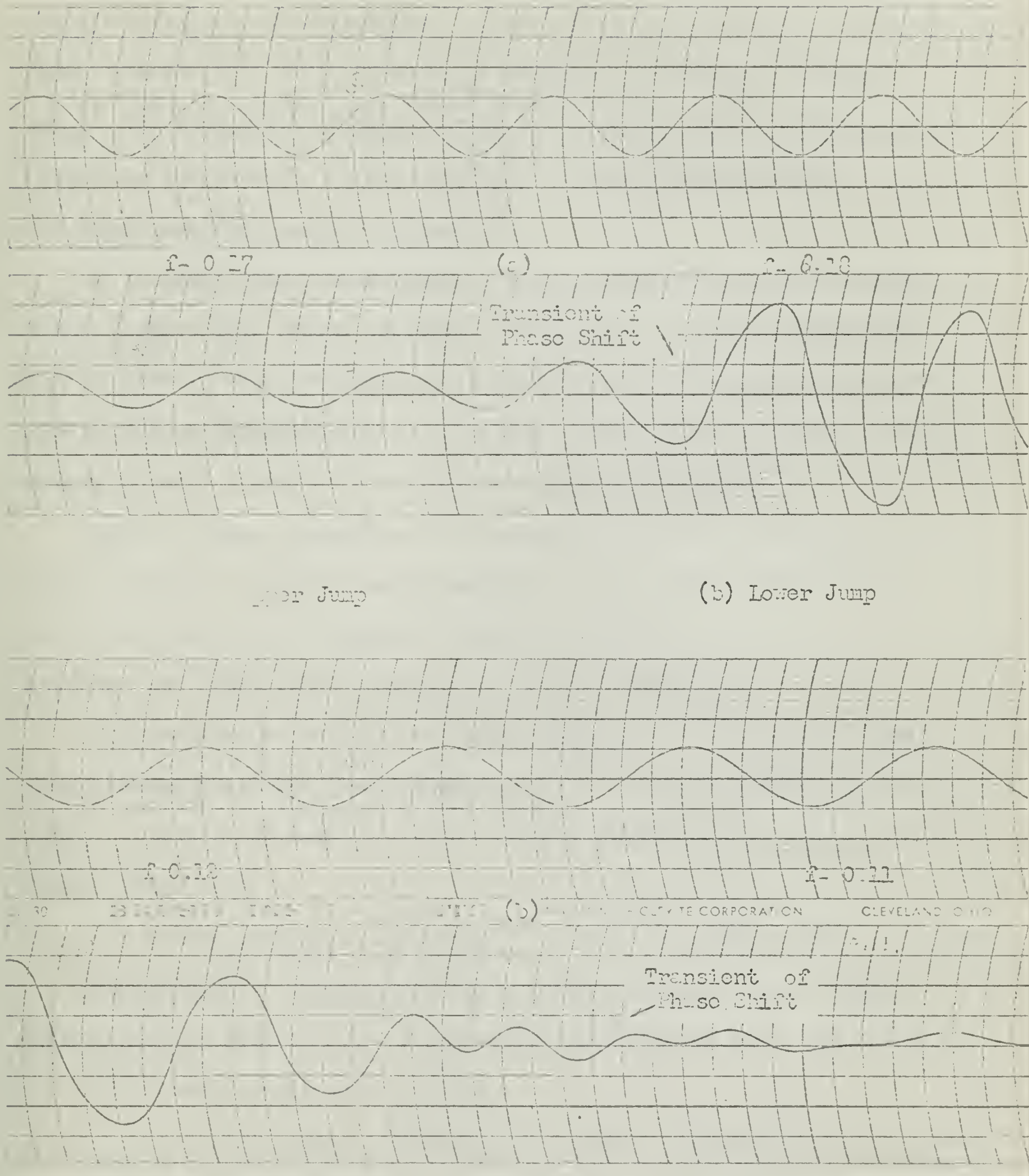


Fig 3 8 Amplitude and Phase Jump
of a Typical System

increase in frequency causes the amplitude to jump upwards from point B to C, and with a phase shift from below 90 degrees to a value above 90 degrees. Further increases in the frequency cause the output to follow the gradual smooth curve to D. Now, if the input frequency is reduced from higher values, the output will follow the smooth curve DC. When the frequency is reduced to the point E, if the frequency is infinitesimal reduced, the output jumps down from point E to point F.

It is found that either from the theory analysis or from the experimental work, a hard spring saturation nonlinearity can also produce the jump resonance, only the difference between them is the jump in the opposite direction from each other. A typical system for a hard spring saturation nonlinearity will be shown in the experimental section.

From the above discussion, the portion of the curve from point B to point E cannot be obtained experimentally, usually this portion of the curve corresponds to an unstable operating condition.

3-6 Lower and Upper Jump Frequency of the Jump Resonance:

As previously discussed, the jump frequency of ω_1 is called "Upper Jump Frequency" and the jump frequency of ω_2 is called "Lower Jump Frequency". Recall the equation (3-41) and the sketch of the response curve shown in Fig. 3-7:

$$(-\omega^2 E_1 + Ka_1E_1 + \frac{3}{4} Ka_3E_1^3)^2 + \alpha^2\omega^2 E_1^2 = \omega^4 F^2 + \alpha^2\omega^2 F^2 \qquad (3\text{-}41)$$

The equation of the locus of the vertical tangents can be found by differentiating equation (3-41) implicitly with respect to E_1 and setting $d\omega\ /dE_1$ equal to zero, the result is:

$$(-\omega^2 + Ka_1 + \frac{3}{4} Ka_3E_1^2)(-\omega^2 + Ka_1 + \frac{9}{4} Ka_3E_1^2) + \alpha^2\omega^2 = 0 \quad (3\text{-}54)$$

Check the equation of (3-54), if the damping is very small, the term $\alpha^2\omega^2$ can be neglected, that leads to a pair of equations:

$$\omega^2 - K(a_1 + \frac{3}{4} a_3E_1^3) = 0 \qquad (3\text{-}55)$$

$$\omega^2 - K(a_1 + \frac{9}{4} a_3E_1^2) = 0 \qquad (3\text{-}56)$$

The curves corresponding to these equations are shown in Fig. 3-9. It is to be noted, the equation of (3-55) is the same as the response curve equation (3-23) for the free oscillation of an undamped system. The equation of (3-56) is the locus of the vertical tangents of curve of Fig. 3-5.

In the case when damping is present, the locus of vertical tangents equation (3-54) will be directed by the equations of (3-55) and (3-56), in particular there should be one branch near the response curve for free oscillation, and another near the points where the curves for the undamped system. In other words, the curve must appear as they are shown in Fig. 3-10.

Rearrange the equation (3-54), and solve for ω, which leads:

$$\omega = \sqrt{\frac{1}{2}(2Ka_1 + 3Ka_3E_1^2 - \alpha^2) \pm \sqrt{(2Ka_1 + 3Ka_3E_1^2 - \alpha^2)^2 - 4K^2(a_1^2 + 3a_1a_3E_1^2 + \frac{27}{16}a_3^2E_1^4)}} \qquad (3\text{-}57)$$

or:

$$\omega = \sqrt{\frac{1}{2}(2\omega_n^2 + 3Ka_3E_1^2 - \alpha^2) \pm \sqrt{(2\omega_n^2 + 3Ka_3E_1^2 - \alpha^2)^2 - 4(\omega_n^4 + 3\omega_n^2Ka_3E_1^2 + \frac{27}{16}K^2a_3^2E_1^4)}} \qquad (3\text{-}58)$$

In which $\omega_n = \sqrt{Ka_1}$, it is the natural frequency of the linear system. If we know the constants of system, and the amplitude of fundamental frequency forced oscillation, the upper and the lower jump frequency can be calculated from either one of the equations (3-54), (3-57), and (3-58).

3-7 Analog Computer Analysis for a Second Order System with Soft Saturation Nonlinearity:

Consider a system with a block diagram shown in Fig. 3-11.

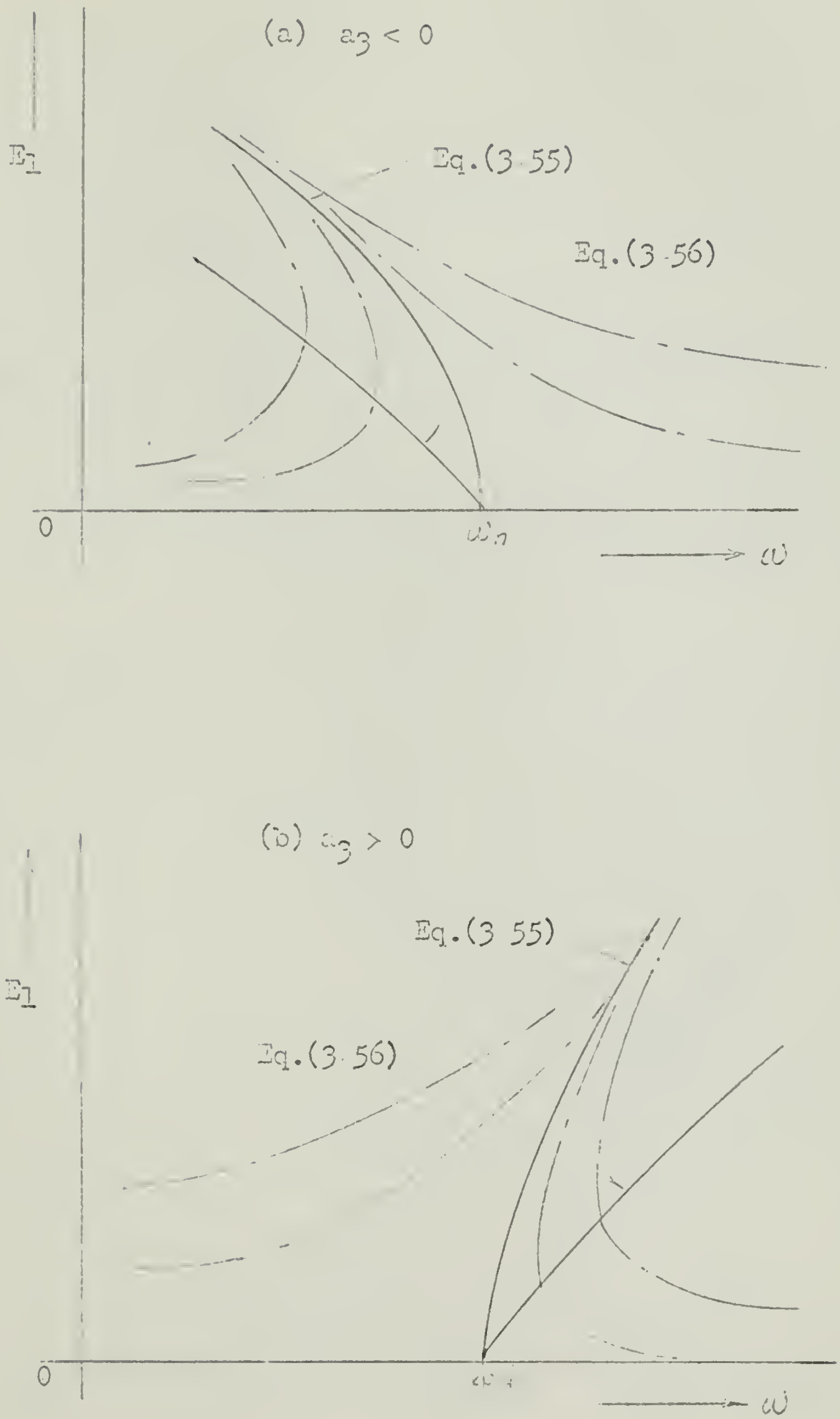


Fig.3.9 Loci of The Vertical Tangents of Response Curve of An Undamping Nonlinear System

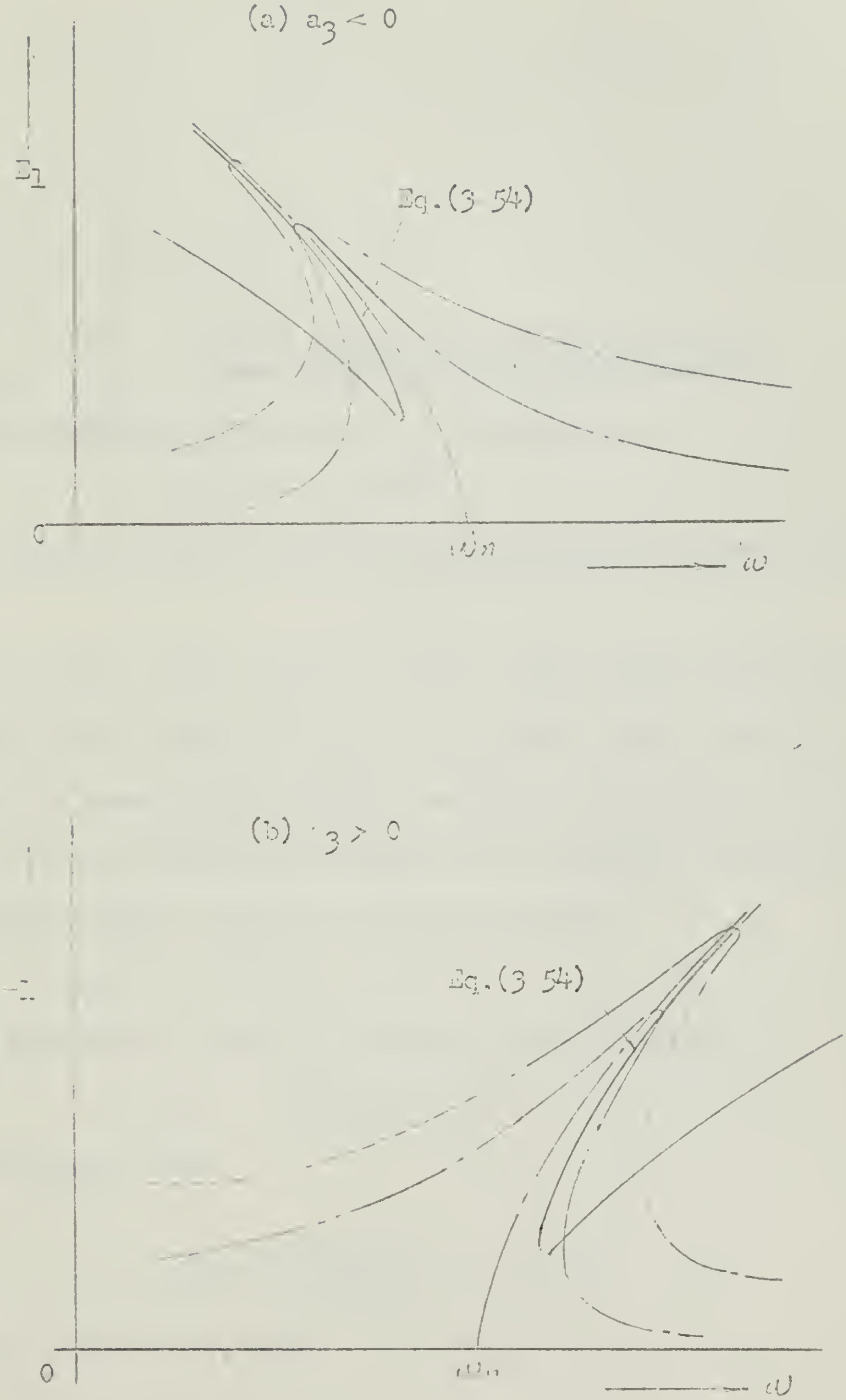


Fig.3 10 Loci of Vertical Tangents of
Response Curves of a Damped Nonlinear System

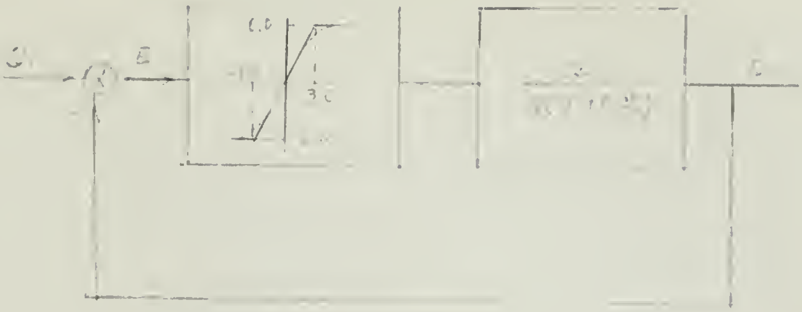


Fig. 3-11 Block Diagram of a 2nd Order System
with a Soft Saturation Nonlinearity

In which the restoring force function of nonlinearity is:

$$f(E) = 2E - 0.032E^3 \tag{3-59}$$

The characteristic curve of restoring force function is drawn in Fig. 3-12 where:

E =	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00
f(E)=	1.97	2.89	3.75	4.50	5.15	5.65	5.95	6.00	5.10

Before the experimental analysis consider the theoretical calculations and curves. First consider the response curve on the E_1-ω plane for the system without damping, and with an input function:

$$\theta_r(t) = 3 \cos(\omega t + \theta) \tag{3-60}$$

Recall the equations of (3-21); (3-22) and (3-23) and leads:

$$\omega = (4 - 0.048E_1^2)^{1/2} \tag{3-61}$$

for free oscillation and:

$$\omega = \left(\frac{4E_1 - 0.048E_1^3}{E + 3}\right)^{1/2} \tag{3-62}$$

for the phase angle $\theta = \pi$, and:

$$\omega = \left(\frac{4E_1 - 0.048E_1^2}{E - 3}\right)^{1/2} \tag{3-63}$$

for the phase angle $\theta = 0$.

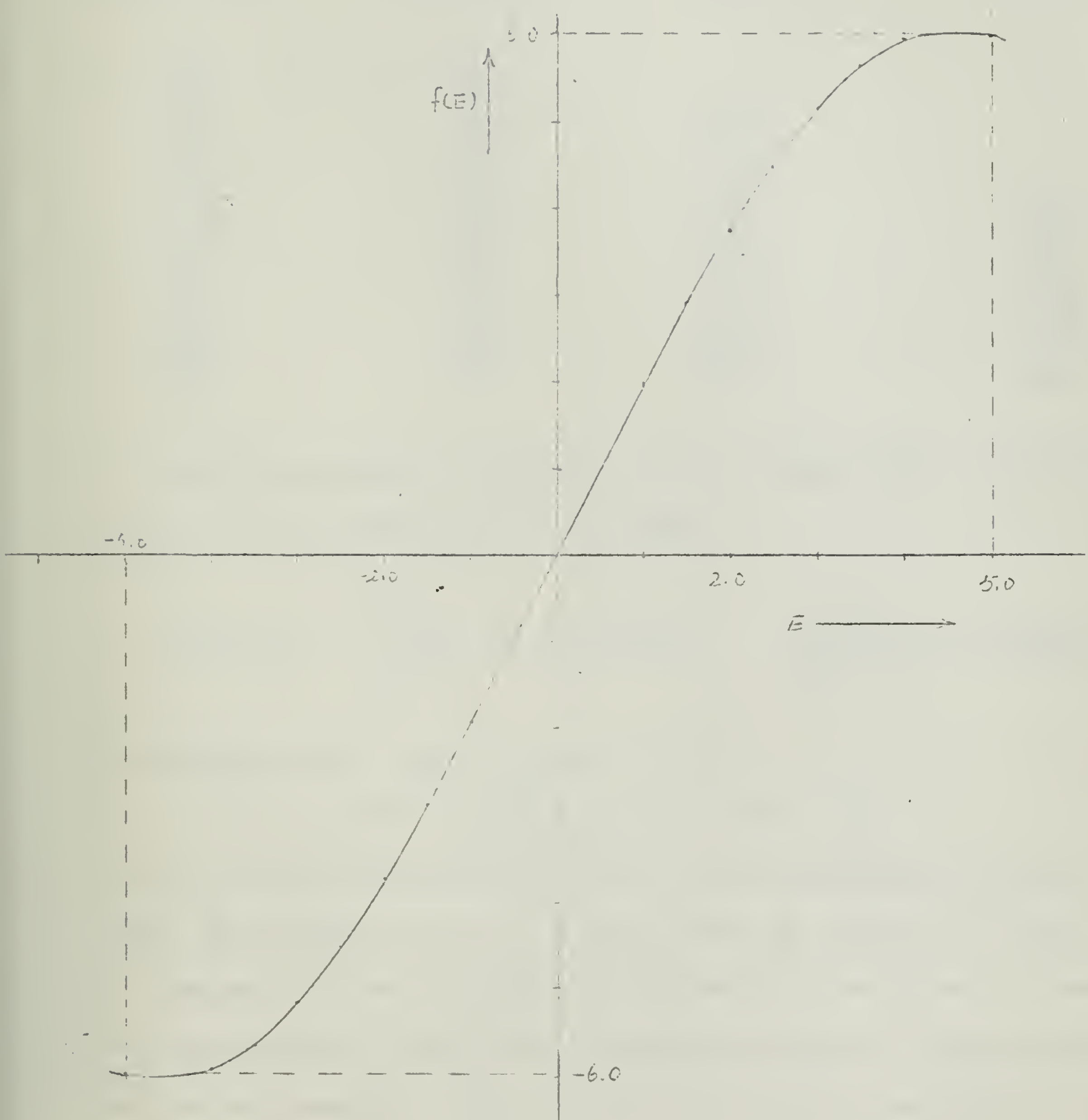


Fig. 3.12 Characteristic Curve of Restoring Function; $f(E) = 2E - 0.032E^3$

The calculations and curves of equation (3-61) to (3-63) are shown in Table 3-1 and Fig. 3-13 respectively.

Table 3-1: Calculation for response curves on the $E_1 \sim \omega$ plane:

Amplitude of Error Signal (E_1)	Frequency for Free Oscill. (ω)	Frequency for $\theta = \pi$ (ω)	Frequency for $\theta = 0$ (ω)
0.00	2.00	/	/
0.50	1.996	0.75	/
1.00	1.99	0.99	/
2.00	1.95	1.23	/
3.00	1.89	1.37	/
3.50	1.84	1.35	4.88
4.00	1.80	1.36	3.60
5.00	1.68	1.32	2.65
6.00	1.53	1.23	2.16
7.00	1.29	1.08	1.73
8.00	1.00	0.80	1.22
9.00	0.35	0.28	0.42

For the calculation of response curves and phase shift, when damping is present, recall equation (3-42), and leads:

$$\omega^4 + \omega^2\left[0.25^2 - \frac{2E_1^2}{E_1^2 - 3^2}(4 - 0.048E_1^2)\right] + \frac{E_1^2}{E_1^2 - 3^2}(4 - 0.048E_1^2)^2 = 0 \qquad (3\text{-}64)$$

for the phase shift, recall equation (3-45):

$$\theta = \mathrm{Sin}^{-1}\frac{E_1 \alpha}{F(\omega^2 + \alpha^2)^{1/2}} + \mathrm{Tan}^{-1}\frac{\alpha}{\omega} \qquad (3\text{-}45)$$

It is to be noted, from equation (3-45), and the response curve for an undamped system shown on the $E_1 \sim \omega$ plane, when the frequency is larger than the frequency of free oscillation with a particular amplitude of input, the phase angle θ is zero and is π, when the frequency is below the free oscillation frequency. Hence; the phase calculation from equation (3-45) should be modified as:

$$\theta = 180 - \left(\mathrm{Sin}^{-1}\frac{0.25E_1}{3(\omega^2 + 0.25^2)^{1/2}} + \mathrm{Tan}^{-1}\frac{0.25}{\omega}\right) \qquad (3\text{-}65)$$

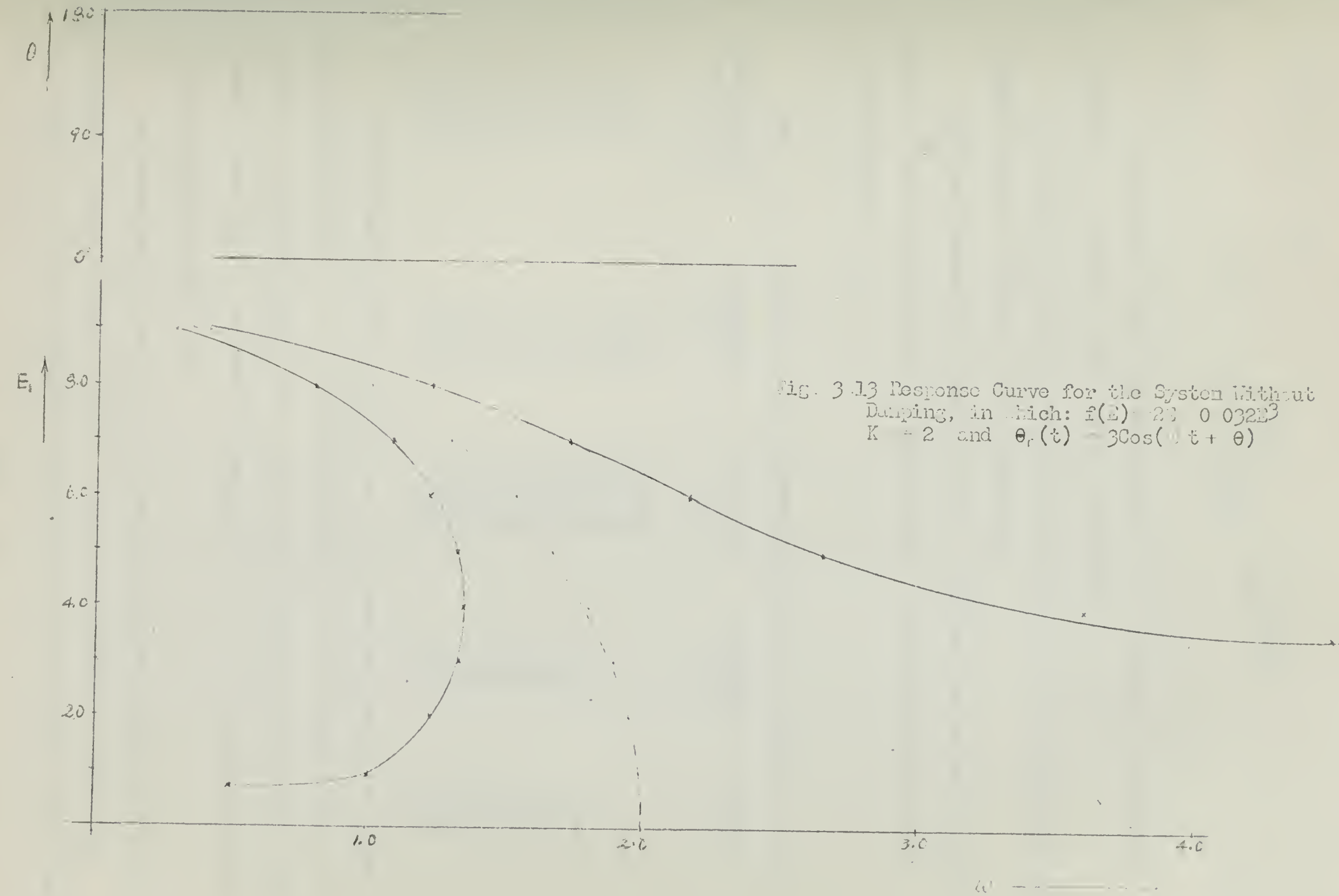


Fig. 3.13 Response Curve for the System Without Damping, in Which: $f(E) = 2E + 0.032E^3$
$K = 2$ and $\theta_r(t) = 3\cos(\omega t + \theta)$

for $\omega < \omega_n$, and:

$$\theta = \mathrm{Sin}^{-1}\frac{0.25E_1}{3(\omega^2 + 0.25^2)^{1/2}} + \mathrm{Tan}^{-1}\frac{0.25}{\omega} \tag{3-66}$$

for $\omega > \omega_n$.

If the response curve is presented on the θ – ω plane, the condition of equation (3-65) and (3-66) should be reversed.

The calculation for the system with damping of 0.25 and the curves are shown in Table 3-2 and Fig. 3-14 respectively.

The jump frequencies of system can be calculated from equation (3-57) and also can be seen from the response curve. It is shown schematically in the diagram of Fig. 3-14.

Table 3-2: Calculation for Response Curves (with a damping of 0.25)

Amplitude of Error Signal (E_1)	Frequency for $\omega < \omega_n$ (ω)	Phase for $\omega < \omega_n$ (θ)	Frequency for $\omega > \omega_n$ (ω)	Phase for $\omega > \omega_n$ (θ)
1.00	1.00	164.00	/	/
2.00	1.24	161.00	/	/
3.00	1.35	158.50	/	/
3.50	1.36	156.50	4.83	3.00
4.00	1.37	155.00	3.58	7.00
4.50	1.40	153.00	2.80	12.00
5.00	1.36	151.50	2.50	15.50
6.00	1.24	144.50	2.05	21.00
7.00	1.09	135.00	1.64	28.50
8.00	0.87	121.50	1.16	45.50
8.70	0.65	90.00	0.65	90.00

For the experimental analysis, the first investigating is the restoring forcing function. It is to be approximated as a two slope lines, one slope is 2, and the other is zero. The circuit setting in the analog computer is shown below, and also all components to be used are schematically in the circuit of Fig. 3-15.

The characteristic curve for the restoring forcing function from the computer is shown in Fig. 3-16; (a) is shown the waveform of input (e_i) and output (e_o), (b) is shown the characteristic of the output to the input.

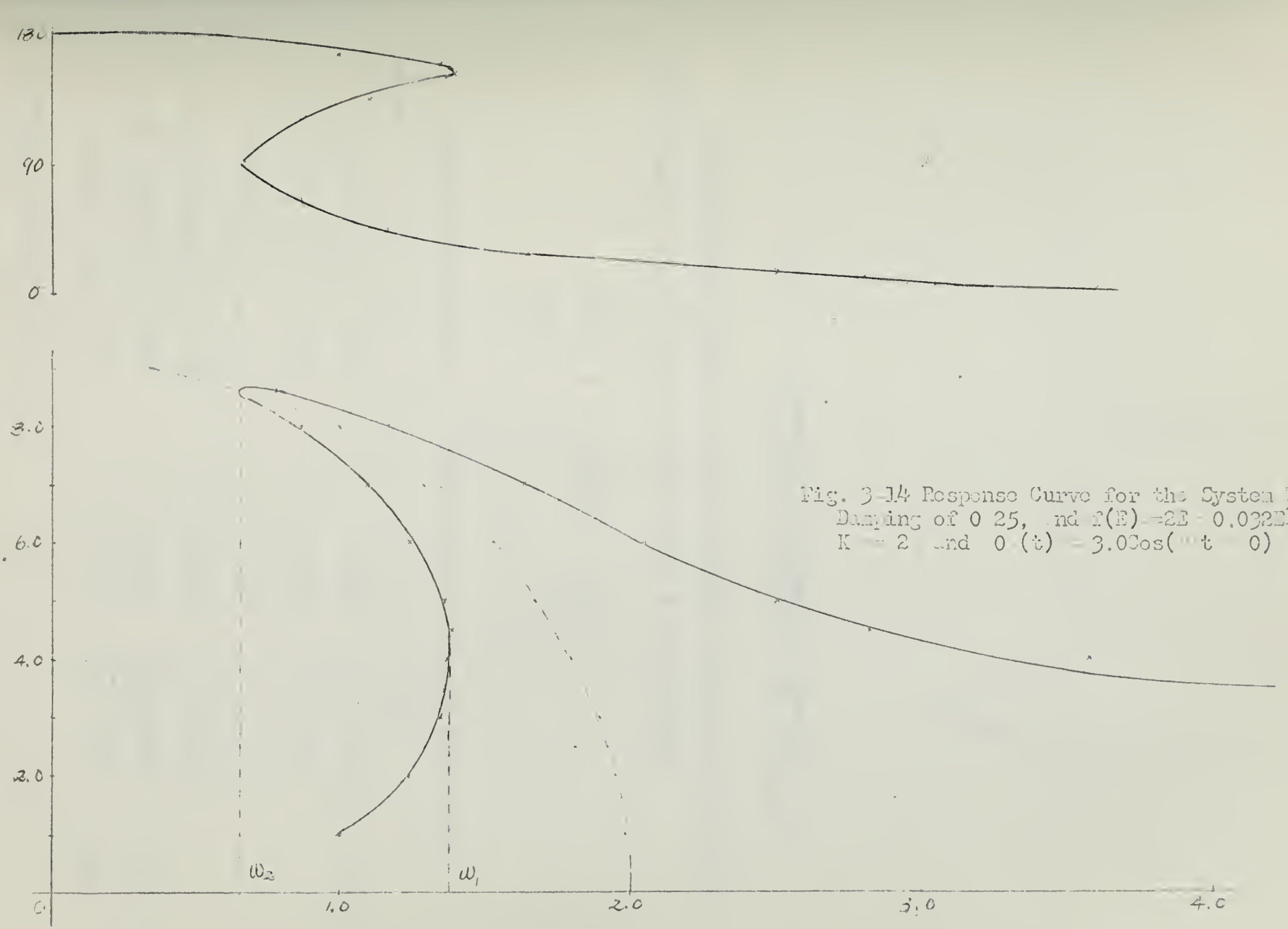


Fig. 3-14 Response Curve for the System With Damping of 0.25, and $f(E) = 2E - 0.032E^3$
$K = 2$ and $\theta(t) = 3.0\cos(\omega t + 0)$

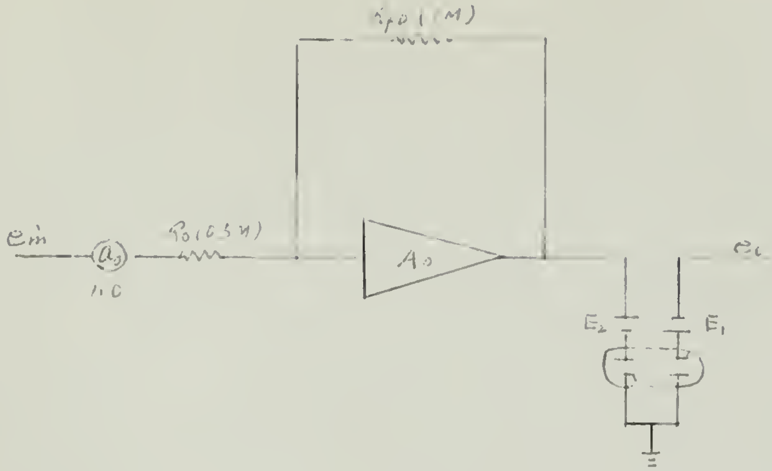


Fig. 3-15 Operational Diagram of a Function Generator for $f(E) = 2E - 0.032E^3$

The operational diagram of analog computer setting up for system of block diagram in Fig. 3-11 is shown in Fig. 3-17, in which:

$$\bar{e}(s) = \frac{\alpha_\theta}{\alpha_e}\left[\bar{\theta}_r(s) - \bar{\theta}_c(s)\right] \tag{3-67}$$

$$\bar{\omega}_c(s) = \frac{3\alpha_e}{\alpha_\omega}\,\frac{1}{4\alpha_t s + 1}\,\bar{e}(s) \tag{3-68}$$

$$\bar{\theta}_c(s) = \frac{\alpha_\omega}{\alpha_\theta \alpha_t}\,\frac{1}{s}\,\bar{\omega}_c(s) \tag{3-69}$$

From equations (3-67) to (3-69) system scaling is shown below:

where: $\alpha_\theta = 1.00$; $\alpha_e = 1.00$; $\alpha_t = 1.00$; $\alpha_\omega = 2.00$

$W_1 = a_1 R_{f1}/R_1 = 1.00$ $a_1 = 1.00$; $R_{f1} = 1.00$; $R_1 = 1.00$

$W_2 = a_2 R_{f1}/R_2 = 1.00$ $a_2 = 1.00$; $R_2 = 1.00$

$W_3 = R_{f2}C_{f1}/a_4 = 4.00$ $a_4 = 0.50$; $R_{f2} = 1.00$

$W_4 = a_3R_{f2}/a_4R_3 = 4.00$ $a_3 = 1.00$; $R_3 = 0.50$;

$W_5 = a_5/R_5C_{f2} = 2.00$ $a_5 = 1.00$; $R_5 = 0.50$ $C_{f2} = 1.00$

$W_6 = a_6R_f/R_6 = 1.00$ $a_6 = 1.00$; $R_6 = 1.00$ $R_{f3} = 1.00$

in which R in megohm; C in μf

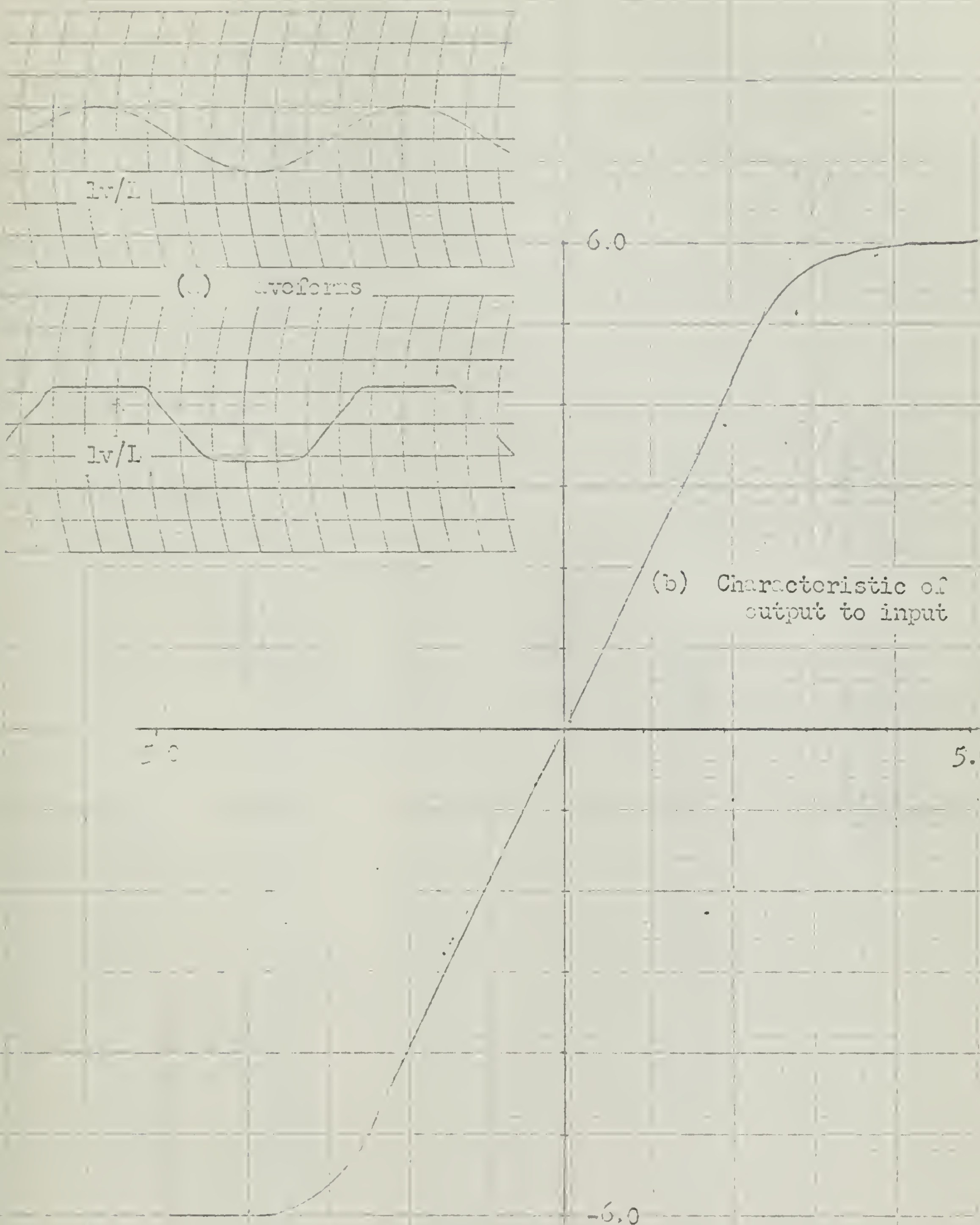


Fig. 3.16 Characteristic curve From Analog Computer for restoring function $f(E) = 2E - 0.032E^3$

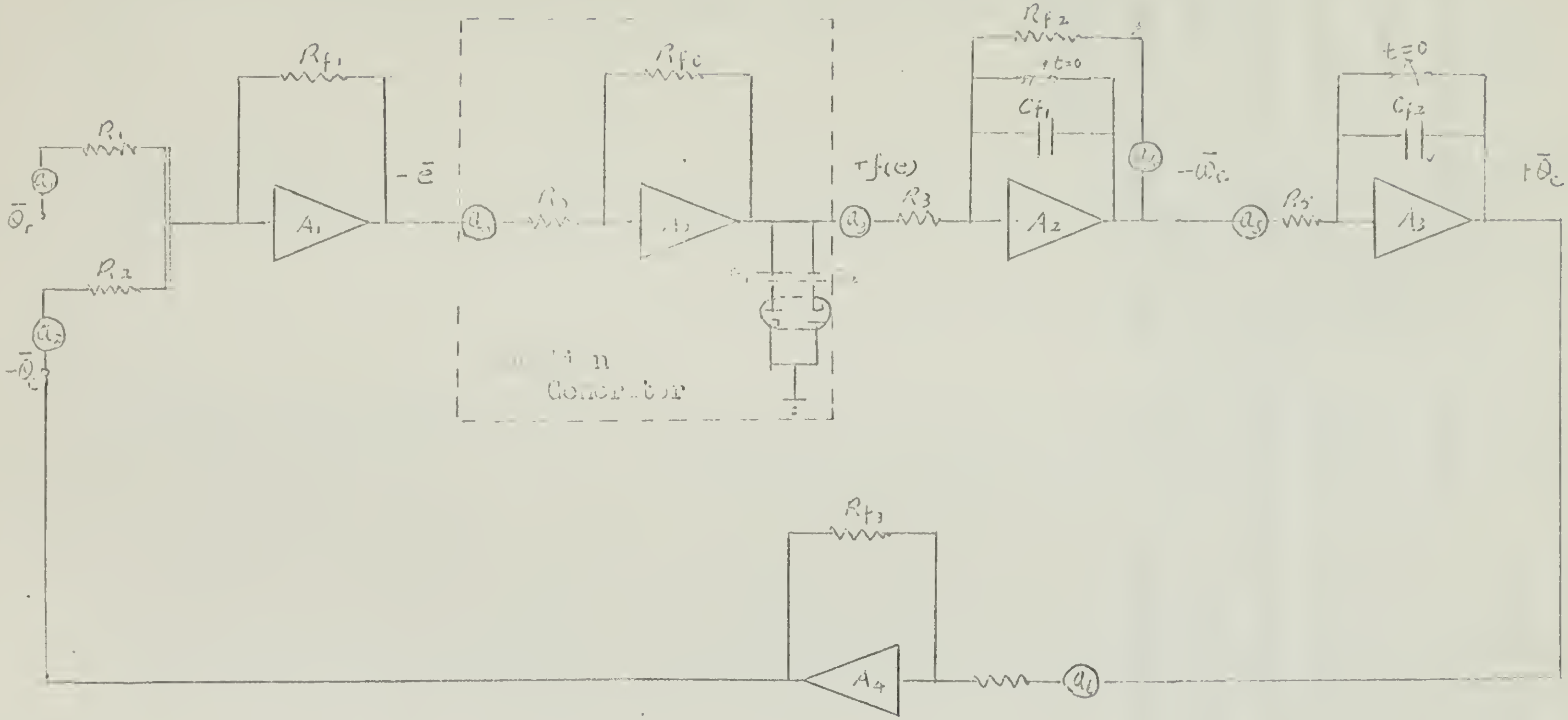


Fig.3-17 Operational Diagram of Analog Computer setting up for System of Block Diagram in Fig 3 14

The results from the analog computer experimental work is condensed in Table 3-3; and the response curve with phase shift represented on E, ψ plane is shown in Fig. 3-18 respectively. Also the jump frequencies are indicated schematically in these diagrams.

A theoretically calculated response curve in E, ψ plane for comparing is shown with the experimental response curve of Fig. 3-18. Comparing the results from the computer experimental work with the results from theoretical calculation, it is shown that the results are very close within high frequency and low frequency range. There is a resonance phenomena between the jump frequencies.

Table 3-3 Data for Response Curves from Analog Computer for System
$\zeta = 0.25$; $K = 2$; $f(E) = 2E - 0.032E^3$: $\theta_r(t) = 3.0\cos(\omega t + \theta)$

(a) Frequency Increasing:

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (ϕ)	Amplitude of Output (C)	Phase of Output (θ)
0.500	0.38	180.00	3.20	0.00
0.628	0.50	180.00	3.40	0.00
0.755	0.75	175.00	3.60	5.00
0.880	1.25	168.00	4.00	9.00
1.000	1.50	163.00	4.50	12.00
1.130	2.15	158.50	5.00	24.00
1.260	3.00	150.00	5.40	35.00
1.260(+)	9.00	/	6.50	/
1.380	8.25	25.00	5.50	140.00
1.510	7.30	14.00	4.50	155.00
1.640	7.00	5.00	3.80	159.00
1.760	6.50	0.00	3.40	165.00
1.890	6.20	0.00	3.00	172.00
2.010	6.00	0.00	2.60	180.00
2.140	5.50	0.00	2.20	180.00
2.260	5.20	0.00	2.00	180.00
2.390	5.00	0.00	1.60	180.00
2.510	4.80	0.00	1.40	180.00
2.830	4.40	0.00	1.20	180.00
3.140	4.00	0.00	1.00	180.00
3.460	3.80	0.00	0.80	180.00
3.770	3.70	0.00	0.60	180.00
4.090	3.60	0.00	0.50	180.00

(b) Frequency Decreasing:

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (ϕ)	Amplitude of Output (C)	Phase of Output (θ)
2.510	4.75	0.00	1.50	180.00
2.200	5.20	0.00	2.00	180.00
1.890	6.20	0.00	3.00	180.00
1.760	6.80	0.00	3.20	173.00
1.640	7.10	0.00	3.80	168.00
1.510	7.58	0.00	4.50	162.00
1.380	8.30	12.00	5.50	156.00
1.260	9.20	23.00	6.80	140.00
1.130	10.00	34.00	8.00	135.00
1.000	11.50	65.00	9.50	120.00
0.880	11.00	/	8.70	/
0.880(-)	0.75	175.00	4.00	0.00

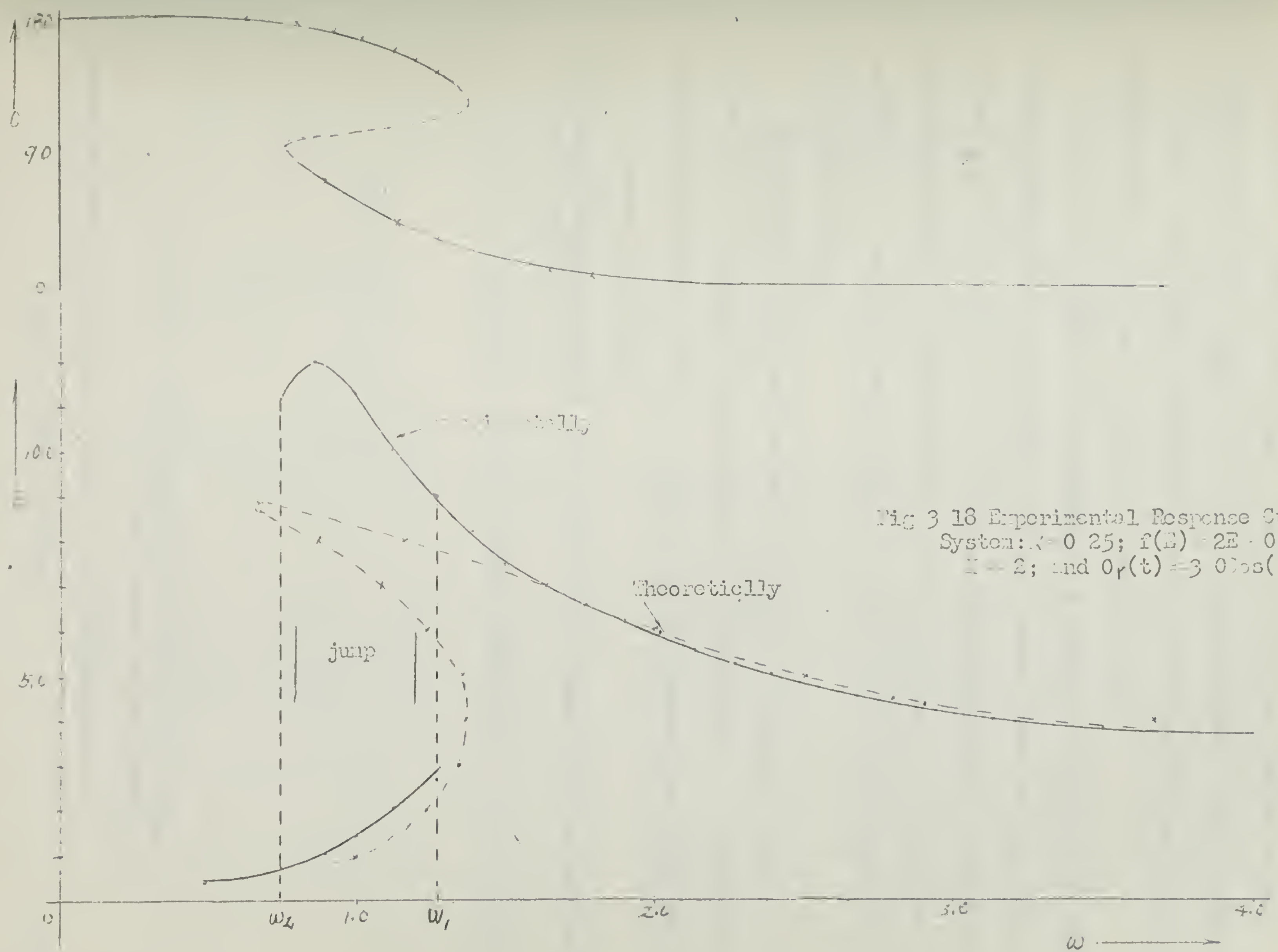


Fig 3 18 Experimental Response Curves for System: ζ = 0 25; f(E) = 2E - 0.032E³
K = 2; and $\theta_r(t) = 3\ 0\cos(\omega t + \theta)$

The resonance phenomena in Fig. 3-18 is actually similar to the resonance in the linear system. It depends on the value of damping in the system. Another experimental result for the same system and same value of input, only changed the damping from 0.25 to 0.5, and is shown in Table 3-4 and Fig. 3-19 respectively. In this case the resonance phenomena is much lower than the case of a damping 0.25.

It is to be noted, if the damping of a system is increased the range of jump frequencies is decreased. For this reason, when the damping increased to limit value, at which no jump phenomena existed, it means that the system becomes a linear system. On the other hand, if the system is an undamped or a very low damping system, it will be very oscillatory or a limit cycle exists.

The response curve represented on "Output vs Frequency" ($\theta_c \sim \omega$) plane can be calculated from equations (3-32) and (3-35). An experimental response curve shown on the $\theta_c \sim \omega$ plane from the system with a damping of 0.25 is shown in Fig. 3-20.

3-8 Analog Computer Analysis for a Second Order System with Hard Saturation Nonlinearity:

Consider a system with a block diagram shown in Fig. 3-21: in which

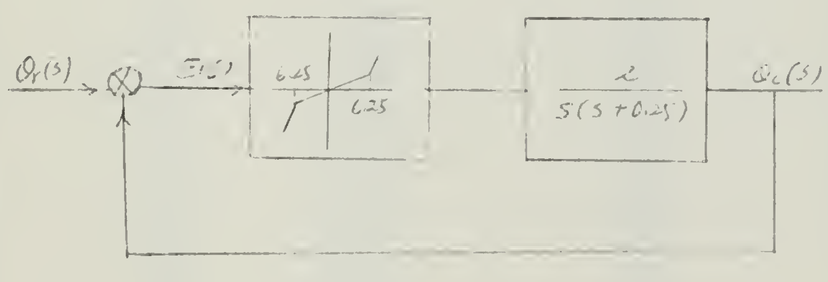


Fig. 3-21 Block diagram of a Second Order System with a Hard Saturation Nonlinearity

assume the restoring forcing function of the nonlinearity is:

$$f(E) = 0.425E + 0.005E^3 \qquad (3\text{-}70)$$

Table 3-4. Data for Response Curve from Analog Computer for System
$\alpha = 0.50$; $K = 2$; $f(E) = 2E - 0.032E^3$; $\theta_r(t) = 3.0\cos(\omega t + \theta)$

(a) Frequency Increasing:

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (θ)	Amplitude of Output (θ_c)	Phase of Output (θ)
0.628	0.50	173.00	3.40	7.30
0.755	0.75	165.00	3.50	12.00
0.880	1.00	155.00	4.00	16.50
1.000	1.50	140.00	4.75	24.00
1.130	1.75	125.00	6.00	30.00
1.260	8.30	91.00	8.00	85.00
1.380	8.10	47.00	6.25	120.00
1.510	7.50	36.00	5.25	135.00
1.640	7.00	25.00	4.25	148.00
1.760	6.50	20.00	3.50	160.00
1.890	6.00	12.00	2.75	172.00
2.010	5.50	5.00	/	/
2.140	5.00	0.00	/	/
2.260	4.50	0.00	1.75	180.00
2.390	4.25	0.00	/	/
2.510	4.00	0.00	1.25	180.00
2.810	3.75	0.00	1.00	180.00
3.140	3.50	0.00	0.75	180.00
3.460	3.35	0.00	0.50	180.00
3.770	3.25	0.00	0.30	180.00
4.090	3.10	0.00	/	/

(b) Frequency Decreasing

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (θ)	Amplitude of Output (θ_c)	Phase of Output (θ)
2.510	4.30	0.00	1.25	180.00
2.200	5.00	0.00	1.75	180.00
1.890	5.75	0.00	2.75	172.00
1.760	6.25	7.50	3.50	160.00
1.640	7.00	15.00	4.10	148.00
1.510	7.75	27.00	5.00	132.00
1.380	8.25	32.00	6.50	120.00
1.260	8.50	60.00	7.50	97.00
1.130	7.00	80.00	8.20	40.00
1.000	2.00	130.00	4.75	19.50
0.880	1.25	145.00	4.00	0.00

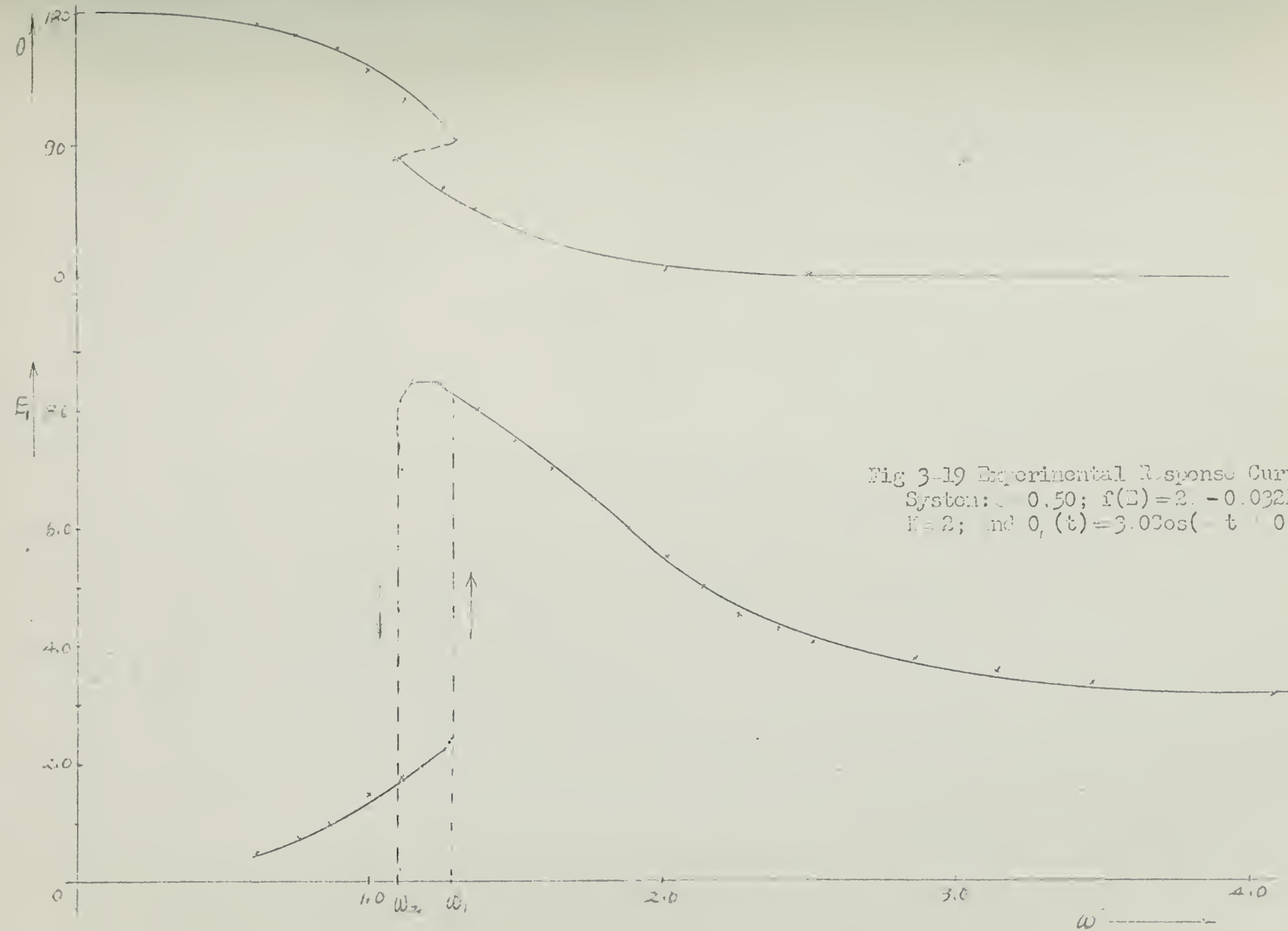


Fig 3-19 Experimental Response Curve for
System: $\zeta = 0.50$; $f(E) = 2E - 0.032E^3$
$K = 2$; and $\theta_i(t) = 3.0\,Cos(\omega t + 0)$

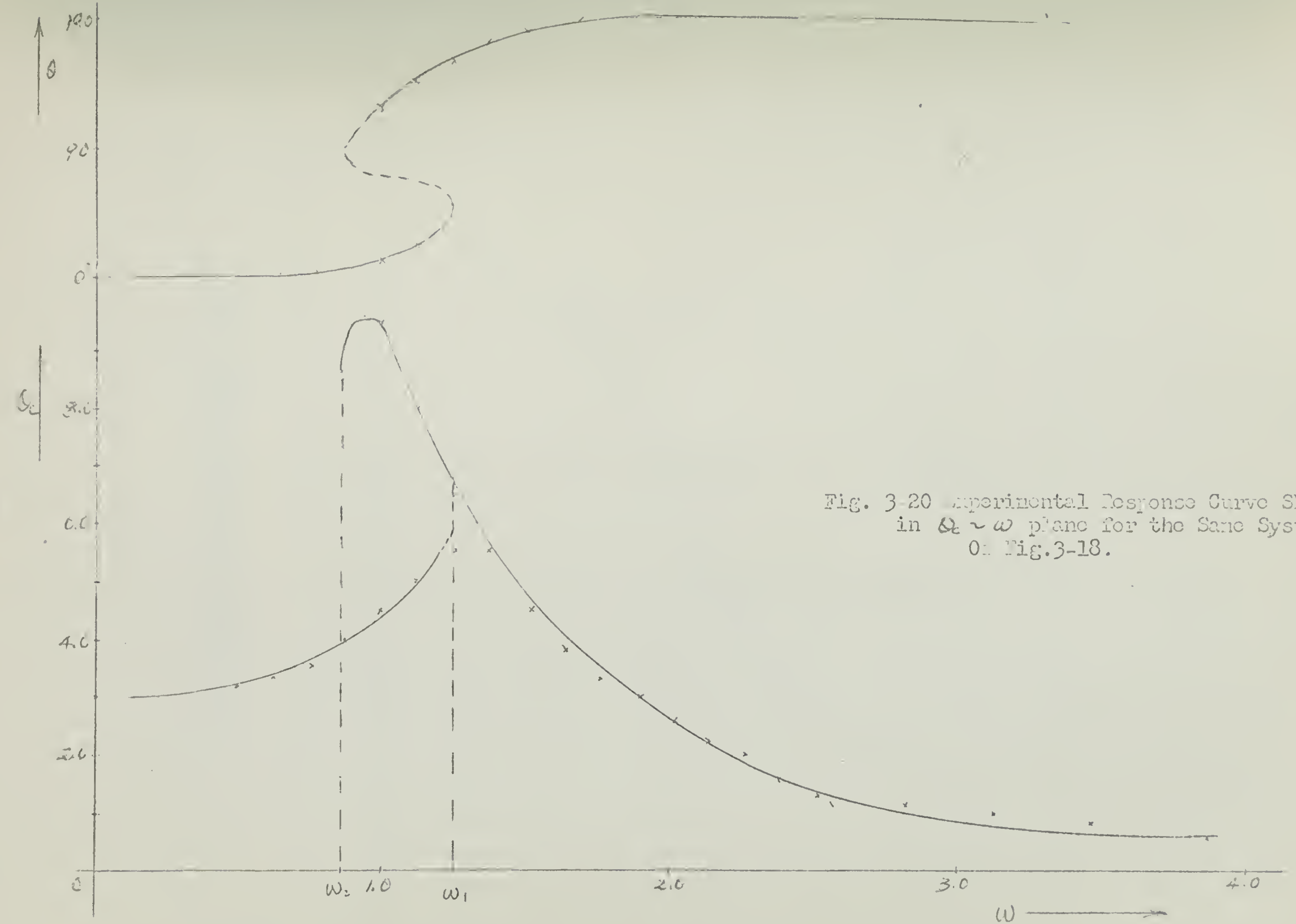


Fig. 3-20 Experimental Response Curve Shown in $\theta_c \sim \omega$ plane for the Same System Of Fig.3-18.

Fig. 3-21: Characteristic Curve of
Restoring function: $f(E) = 0.425E + 0.005E^3$

The characteristic curve of equation (3-70) is shown in Fig. 3-21a, where:

$E = 1.00;\quad 2.00;\quad 3.00;\quad 4.00;\quad 5.00;\quad 6.00;\quad 7.00;\quad 8.00.$

$f(E) = 0.43;\quad 0.89;\quad 1.42;\quad 2.02;\quad 2.75;\quad 3.63;\quad 4.69;\quad 5.96.$

A theoretical analysis can be calculated from equations (3-70) and (3-21) to (3-23), as we did in section 3-7. Only the analog computer analysis is considered in this section.

For the simulation of the restoring forcing function of nonlinearity from the analog computer, it is approximated by two straight lines, one with a slope of 0.425, and the other with a slope of 2.00. The circuit setting in the computer is shown in Fig. 3-22, and all components to be used are schematically in the diagram.

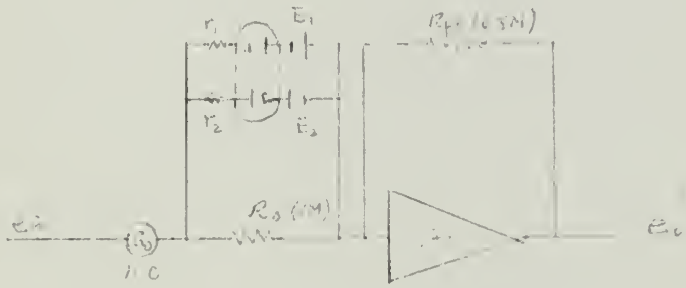


Fig. 3-22 Operational diagram of the Function Generator for $f(E) = 0.425E + 0.005E^3$

The characteristic curve of restoring forcing function from the computer is shown in Fig. 3-23.

The operational diagram of analog computer set up for the whole system is mostly the same with the diagram of Fig. 3-17, the only difference is the function generator circuit.

Fig. 3-23 Characteristic Curve From The Analog Computer for: $f(e) = 0.425e + 0.005e^3$

The results from the analog computer are condensed in Table 3-5, and the response curves with phase shift shown on the [illegible] and [illegible] plane are shown in Fig. 3-24 and 3-25 respectively. In which the input forcing function is $\theta\ (t) = 3.50\ \mathrm{Cos}(\omega t + \theta)$.

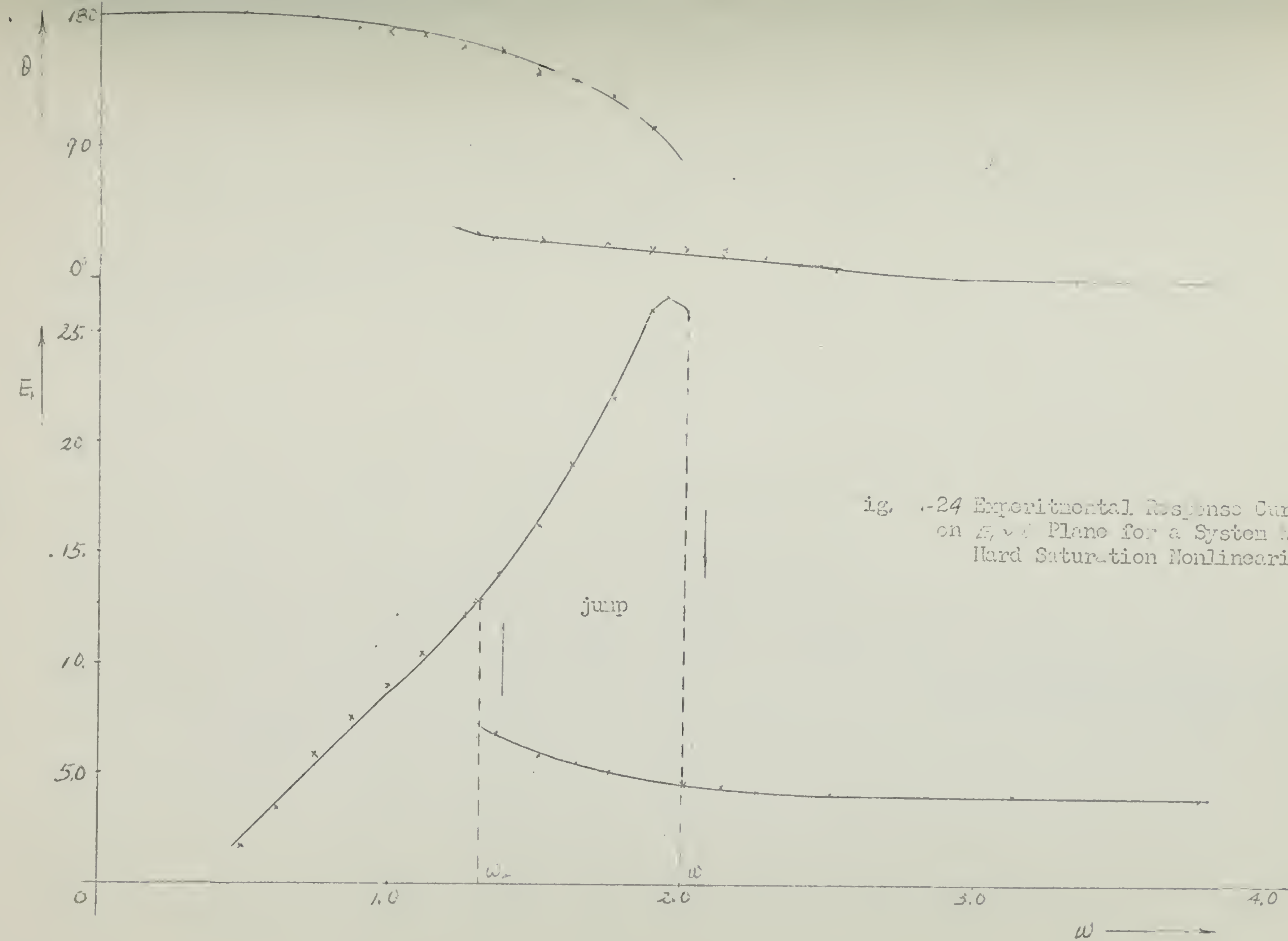
Table 3-5. Data for Response Curves from Computer for a System with a Hard Saturation Nonlinearity, in which: α = 0.25; K = 2; $f(E) = 0.425E + 0.005E^3$; $\theta_r(t) = 3.5 \, Cos(\omega t + \theta)$.

(a) Frequency Increasing:

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (θ)	Amplitude of Output (θ_c)	Phase of Output (θ)
0.500	1.75	180.00	5.50	0.00
0.628	3.50	180.00	6.50	4.00
0.754	6.50	175.00	9.00	7.50
0.880	8.00	168.00	10.50	10.00
1.000	9.00	164.00	12.00	11.50
1.130	10.50	158.50	13.00	13.00
1.260	12.25	152.00	14.50	15.00
1.380	14.00	145.50	16.30	17.00
1.510	16.25	141.00	18.75	20.00
1.640	19.00	134.00	21.30	24.00
1.760	22.25	124.00	22.00	37.00
1.890	26.00	102.00	25.00	45.00
1.950	26.50	/	25.50	55.00
2.010	26.00	24.00	25.00	90.00
2.140	4.50	20.00	0.86	150.00
2.260	4.35	15.00	0.65	165.00
2.510	4.25	7.50	0.50	180.00
2.830	4.15	0.00	0.40	180.00
3.140	4.00	0.00	0.30	180.00
3.770	3.80	0.00	/	/

(b) Frequency Decreasing:

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (θ)	Amplitude of Output (θ_c)	Phase of Output (θ)
2.510	4.25	5.00	0.50	180.00
2.400	4.35	10.50	0.63	180.00
2.270	4.50	13.50	0.68	172.50
2.140	4.60	15.00	0.75	170.00
1.890	4.85	21.00	1.13	157.50
1.760	5.00	23.50	1.25	155.00
1.640	5.35	25.00	1.60	152.00
1.510	5.75	26.00	2.13	150.00
1.380	6.75	27.50	16.30	18.50
1.320	12.25	/	/	/
1.260	11.70	150.00	14.50	15.50
1.130	10.25	155.00	13.00	13.50
1.000	9.00	160.00	11.70	12.00
0.880	7.50	172.00	10.50	10.00



ig. -24 Experimental Response Curve Shown on E, ω Plane for a System With a Hard Saturation Nonlinearity

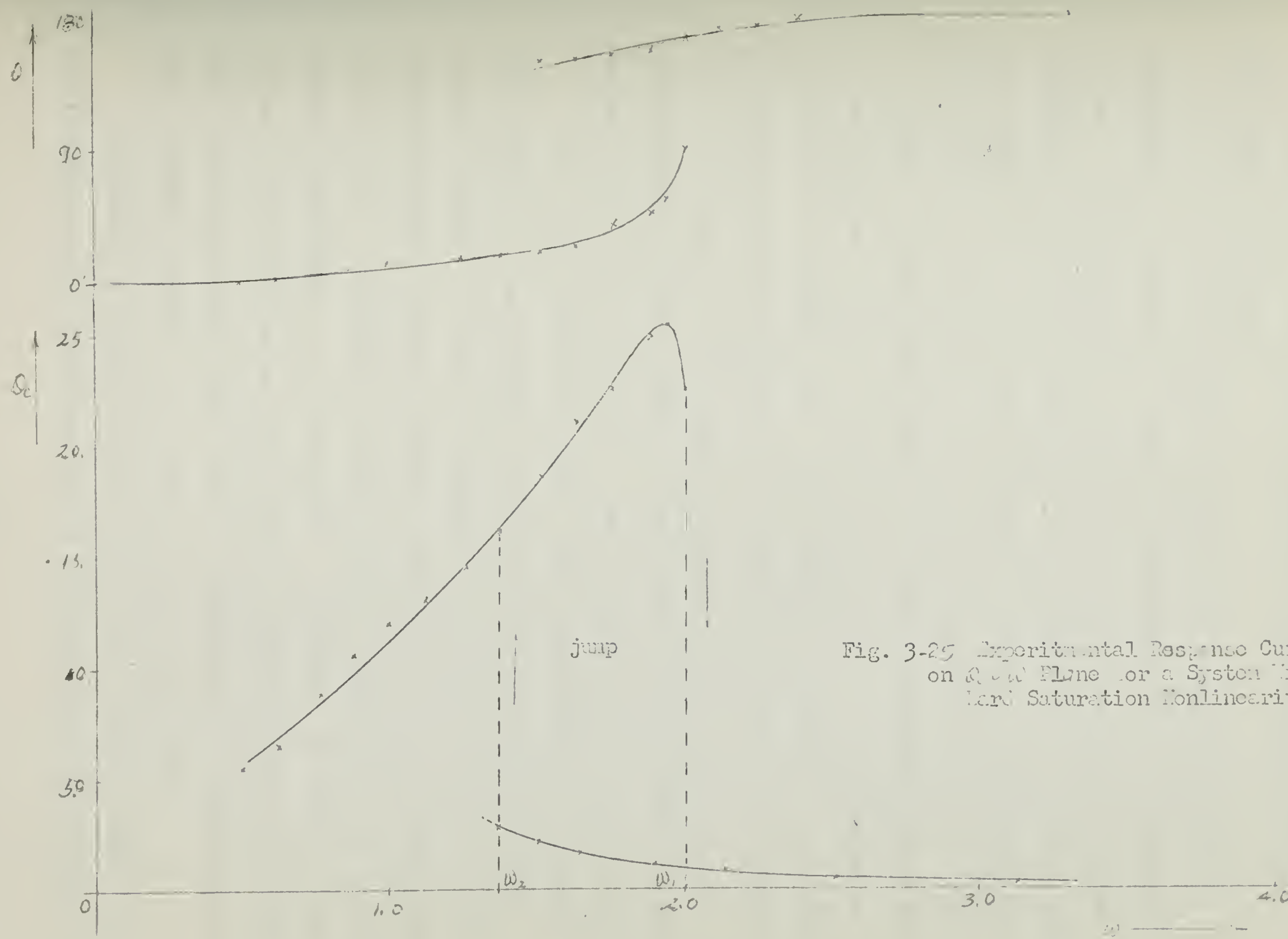


Fig. 3-25 Experimental Response Curve Shown on Q-ω Plane for a System With a Hard Saturation Nonlinearity

CHAPTER IV

SUB-HARMONIC OSCILLATION IN NONLINEAR FEEDBACK CONTROL SYSTEM

4-1 General Description:

Up to the preceding chapter, there is only discussed the forced oscillations for which the frequency is the same as that of the external force. A nonlinear control system, under suitable conditions may have a steady state oscillation in which the main component has a frequency which is a fraction 1/n (n is a real and positive integer) of the forcing function frequency. These oscillations are called "Sub-Harmonic Oscillation".

Recall a linear system (Section 2-3), there are two oscillation terms in the transient response, if the frequency of the free oscillation is ω/n (n is an integer), then a force function of frequency ω can excite the free oscillation in addition to the forced oscillation of frequency ω. But as t increases, $e^{-\zeta t/2}$ damps the free oscillation until only the steady state oscillation remains. Hence there is no sub-harmonic or super-harmonic oscillation existing in a linear control system at steady state.

In the case of nonlinear control system, the steady state response may not have the same frequency of oscillation as the frequency of the forcing function. In fact, the frequency of the system response depends not only upon the frequency of forcing function but also upon the amplitude of the input. The principle of superposition no longer holds and the linear circuit characteristic is not applicable. In general, it can be said that the nonlinear system can have wide variety of almost periodic oscillations, the frequencies of which differ from the frequency of the applied force and vary with time as well as with different initial conditions.

Sub-harmonic oscillation always existed with a frequency higher than the frequency of free oscillation with a particular amplitude of error signal.

The response curve which is a part of an ellipse or a hyperbola in the $E_1 \sim \omega$ plane is always nearly parallel to the right side of the free oscillation response curve. A typical sketch of a soft saturation nonlinearity is shown in Fig. 4-1.

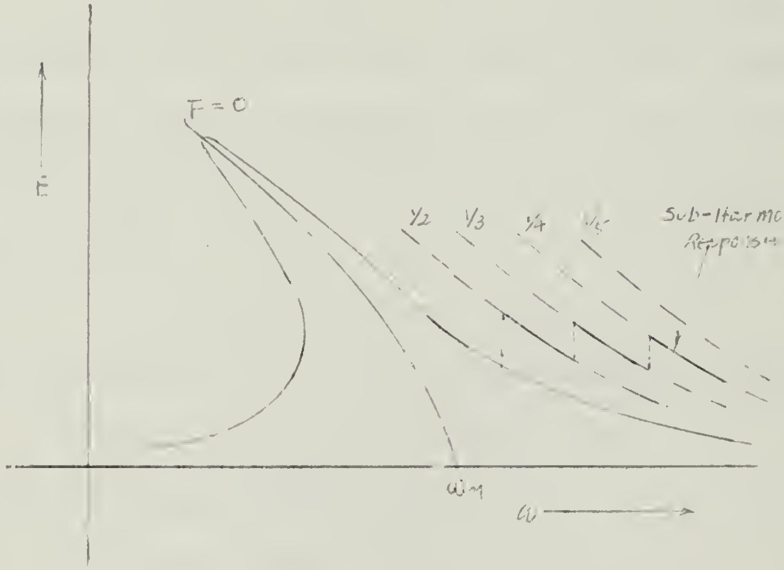


Fig. 4-1 Fundamental and Sub-harmonic Response Curves for a Soft Saturation Nonlinearity

Under some suitable conditions, there may exist any order of sub-harmonic oscillation, or only one type of order can exist depending on the characteristic of the nonlinearity and the input forcing function.

Sub harmonic oscillation and response of nonlinear vibratory systems are often discussed in the field of mechanical system [6,7]. There is no body of knowledge applied to feedback control servomechanism. For the feedback control system, there is a standard differential equation (3-6a), it is very similar to the equation for the mechanical system, the only difference is that the forcing function of a feedback control system is a function of input frequency.

Although any order of sub-harmonic oscillation can exist in a nonlinear feedback control system, in this paper only the sub-harmonic of order

2 and 3 are discussed.

4-2 Sub-harmonic oscillation of Order 2 without Damping:

Sub-harmonic oscillation of order 2 can exist in most nonlinear feedback controls. When it occurs, the frequency of output (θ_c) or error (e) is only 1/2 the frequency of input (θ_r). A typical diagram from the computer experimental work shown on the "Input vs Output" plane and the waveforms of "Input and Output" are shown in Fig. 4-2 and 4-3, respectively.

Recall the basic equation for forced oscillation in nonlinear system with un-damping Eq. (3-11).

$$\ddot{E} + Kf(E) = -A \text{ Cos}(\omega t + \theta) \qquad (3\text{-}11)$$

For the same type of restoring force function with chapter 3, leads:

$$\ddot{E} + Ka_1E + Ka_3E^3 = - A \text{ Cos}(\omega t + \theta) \qquad (3\text{-}12)$$

Assume one solution of equation (3-12) is:

$$E = E_2 \text{Cos} \frac{1}{2}\omega t + E_1 \text{ Cos } \omega t \qquad (4\text{-}1)$$

Where $E_{1/2}$ is the amplitude of sub-harmonic of order 2, E_1 is the amplitude of fundamental frequency, since:

$$\ddot{E} = \frac{1}{4} \omega^2 E_{1/2} \text{ Cos } \frac{1}{2}\omega t - \omega^2 E_1 \text{ Cos } \omega t \qquad (4\text{-}2)$$

$$E^3 = \frac{3}{4}E_{1/2}E_1 + (\frac{3}{4}E_{1/2}^3 + \frac{3}{2}E_{1/2}E_1^2) \text{Cos} \frac{1}{2}\omega t + (\frac{3}{2}E_{1/2}^2E_1 + \frac{3}{4}E_1^3) \text{Cos} \omega t$$

$$+ \frac{1}{4}(E_{1/2}^3 + \frac{3}{4}E_{1/2}E_1^2) \text{Cos} \frac{3}{2}\omega t + \frac{3}{4}E_{1/2} \text{Cos} 2\omega t$$

$$+ \frac{3}{4}E_{1/2}E_1^2 \text{Cos} \frac{5}{2}\omega t + \frac{1}{4}E_1^3 \text{Cos} 3\omega t \qquad (4\text{-}3)$$

We are only interested to the terms of second sub-harmonic and fundamental frequency, inserting equations (4-1) to (4-3) into equation (3-12) the result is:

$$(-\frac{\omega^2}{4}E_{1/2} + Ka_1E_{1/2} + \frac{3}{4}Ka_3E_{1/2}^3 + \frac{3}{2}Ka_3E_{1/2}E_1^2) \text{Cos} \frac{1}{2}\omega t + (-\omega^2E_1 + Ka_1E_1$$

$$+ \frac{3}{2}Ka_3E_{1/2}^2E_1 + \frac{3}{4}Ka_3E_1^3) \text{Cos} \omega t = -A(\text{Cos} \omega t \text{ Cos} \theta - \text{Sin} \omega t \text{ Sin} \theta) \qquad (4\text{-}4)$$

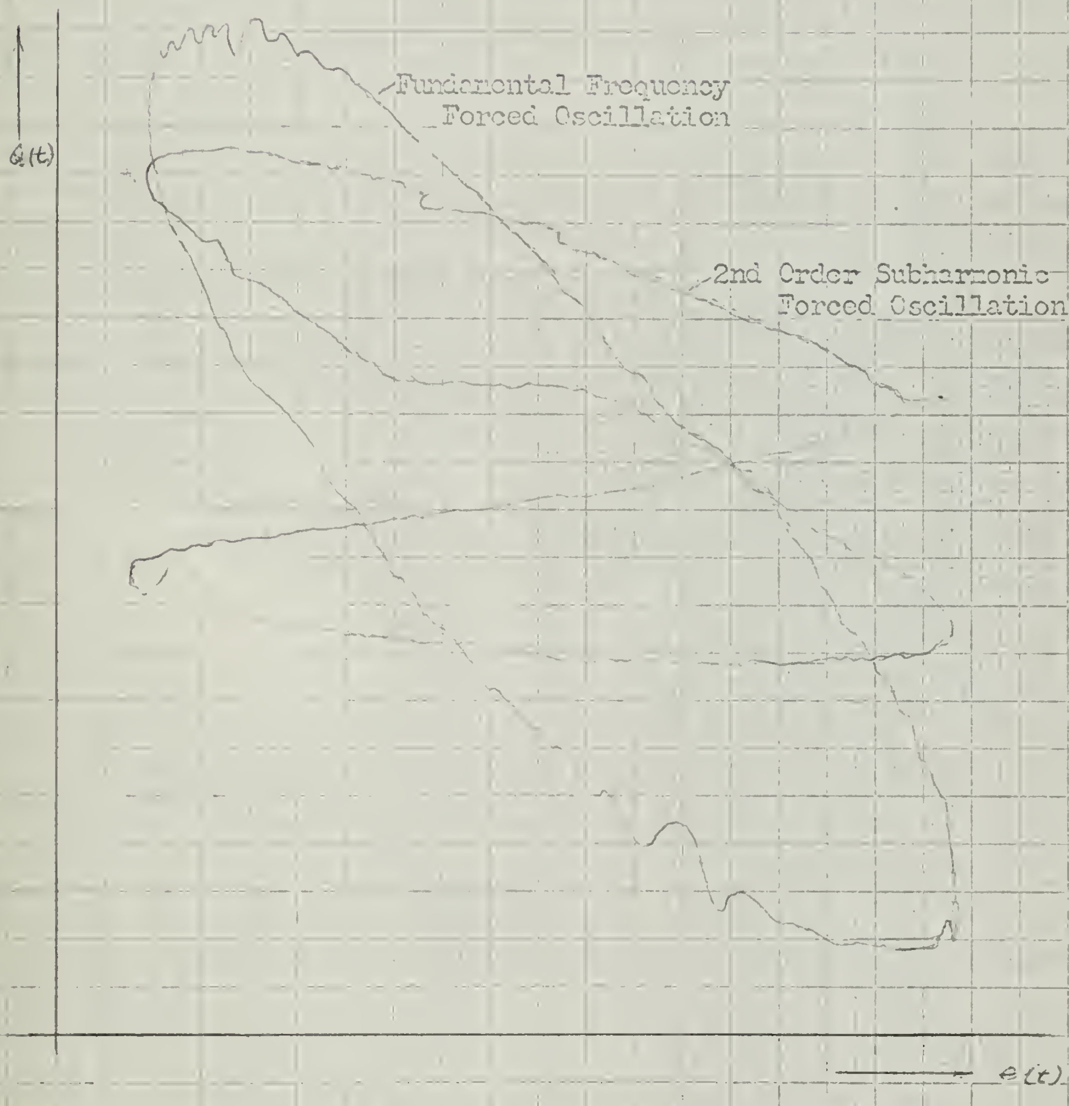


Fig. 4-2 2nd Order Subharmonic Forced Oscillation Shown in The "Output vs Input Plane

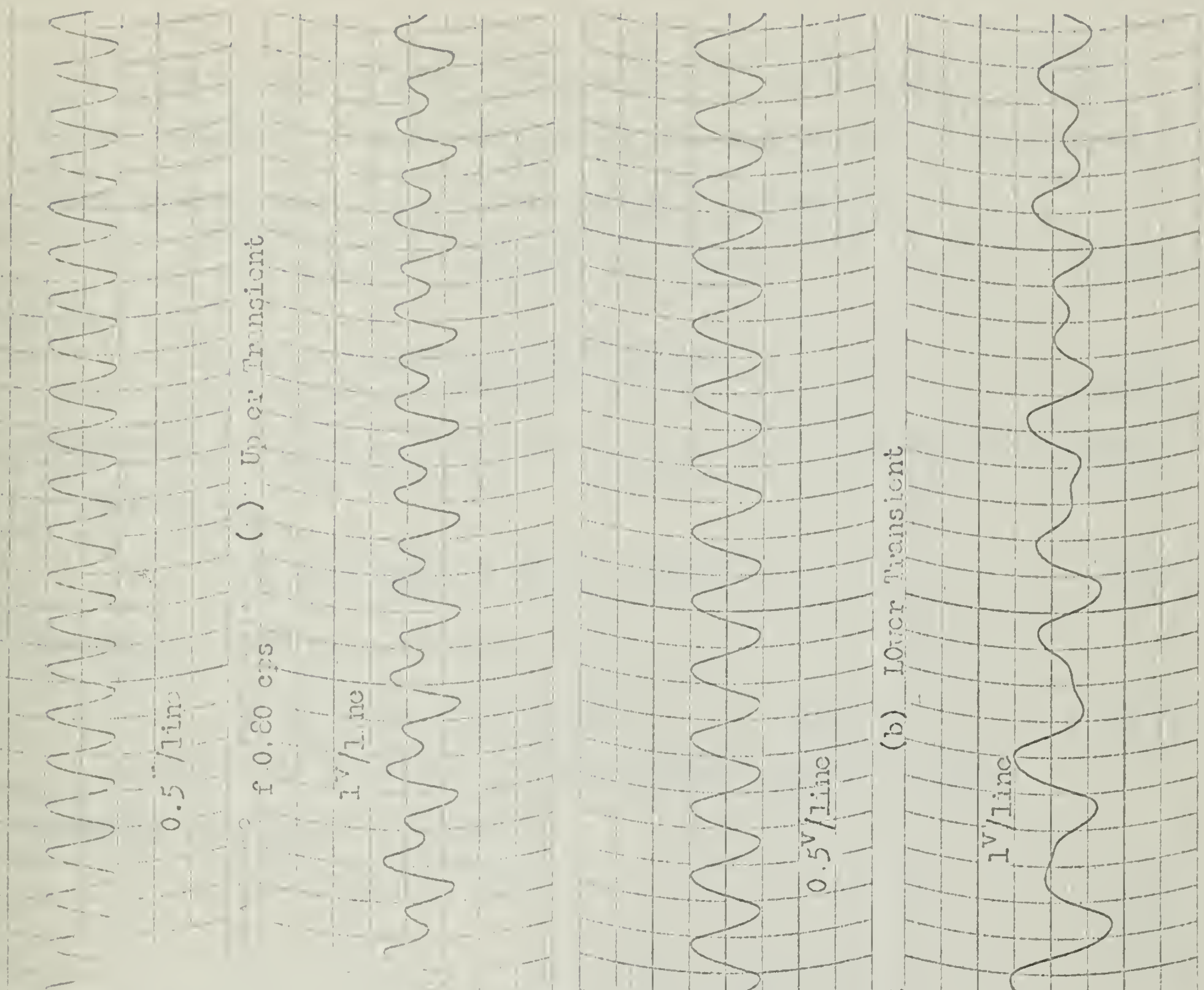


Fig4-3 2nd Order Sub-harmonic Waveforms
for a System: $f(E) = 2E - 0.0048E^3$
$K = 4; \quad \alpha = 0.25$
$\theta_r(t) = 2.50\mathrm{Cos}(\omega t + 0)$

As discussed in Chapter 3, Section 3-3, if the frequency is larger then the frequency of free oscillation, the phase angle θ is zero on the "Error vs Frequency" plane, hence equation (4-4) leads:

$$\omega^2 = 4Ka_1 + 3Ka_3E_{1/2}^2 + 6Ka_3E_1^2 \tag{4-5}$$

$$\omega^2 = \frac{[illegible]\,E_1 + [illegible]\,Ka_3E_{1/2}^2 + \frac{3}{4}Ka_3E_1^3}{E_1 - F} \tag{4-6}$$

Equation (4-5) is the condition for existence of the sub-harmonic of order 2. Solve $E_{1/2}$:

$$E_{1/2} = \pm\left(\frac{[illegible]}{3Ka_3} - \frac{4}{3}\frac{a_1}{a_3} + 6E_1^2\right)^{1/2} \tag{4-7}$$

Since $E_{1/2}$ should be real; which leads:

$$\omega \gtrless (4Ka_1 + 18Ka_3E_1^2)^{1/2} \qquad a_3 \lessgtr 0 \tag{4-8}$$

For the first iteration method, consider a_3 equal to zero: Equations (4-5) and (4-6) leads:

$$\omega^2 = 4Ka_1 = 4\omega_1 \tag{4-9}$$

$$E_1 = \frac{4}{3}F \tag{4-10}$$

Inserting equations (4-9) and (4-10) into equation (4-8):

$$\omega \gtrless 2(\omega_1 + 8Ka_3F^2)^{1/2} \qquad a_3 \lessgtr 0 \tag{4-11}$$

From Equation (4-5), it represents an ellipse or a phperbola in the $E_{1/2}$ vs ω plane depending on the sign of a a_3, also ω has a maximum when a_3 is less than zero, and a minimum when a_3 is greater than zero. Actually, the sub-harmonic oscillation only existed within an interval of frequency. That means only one part of ellipse or hyperbola can exist, as shown in Fig. (4-4).

4-3 Sub-harmonic Oscillation of Order 2 with Damping:

Recall the basic equation (3-41) for a damped system:

$$\ddot{E} + \alpha\dot{E} + Ka_1E + Ka_3E^3 = -A\cos(\omega t + \theta) - B\sin(\omega t + \theta) \tag{3-41}$$

Assume one solution of equation (3-41) is:

$$E = E_{1/2}\cos\frac{1}{2}\omega t + E_{1A}\cos\omega t + E_{1B}\sin\omega t \qquad (4\text{-}12)$$

Since:

$$\dot{E} = -\frac{1}{2}E_{1/2}\,\omega\sin\frac{1}{2}\omega t - E_{1A}\,\omega\sin\omega t + E_{1B}\cos\omega t \qquad (4\text{-}13)$$

$$\ddot{E} = -\frac{1}{4}\omega^2 E_{1/2}\cos\frac{1}{2}\omega t - E_{1A}\,\omega^2\cos\omega t - E_{1B}\sin\omega t \qquad (4\text{-}14)$$

$$\begin{aligned} E^3 = {} & \left(\frac{3}{4}E_{1/2}^3 + \frac{3}{2}E_{1A}^2E_{1/2} + \frac{3}{2}E_{1B}^2E_{1/2}\right)\cos\frac{1}{2}\omega t \\ & + \left(\frac{3}{4}E_{1A}^3 + \frac{3}{4}E_{1B}^3 + \frac{3}{2}E_{1/2}^2E_{1A}\right)\cos\omega t \\ & + \left(\frac{3}{4}E_{1B}^3 + \frac{3}{4}E_{1A}^3 + \frac{3}{2}E_{1/2}^2E_{1B}\right)\sin\omega t \end{aligned} \qquad (4\text{-}15)$$

Inserting equations (4-12) to (4-15) into equation (3-41) and equaling the coefficients of Cos $\frac{1}{2}\omega t$, Cosωt, and Sinωt, give a result of

$$-\frac{1}{4}\omega^2E_{1/2} + Ka_1E_{1/2} + Ka_3E_{1/2}\left(\frac{3}{4}E_{1/2}^2 + \frac{3}{2}E_{1A}^2 + \frac{3}{2}E_{1B}^2\right) = 0 \qquad (4\text{-}16)$$

$$\begin{aligned} -E_{1A}\omega^2 + \alpha E_{1B}\,\omega + Ka_1E_{1A} + Ka_3\left(\frac{3}{4}E_{1A}^3 + \frac{3}{4}E_{1B}^3 + \frac{3}{2}E_{1/2}^2E_{1A}\right) & \\ = -A\cos\theta - B\sin\theta & \end{aligned} \qquad (4\text{-}17)$$

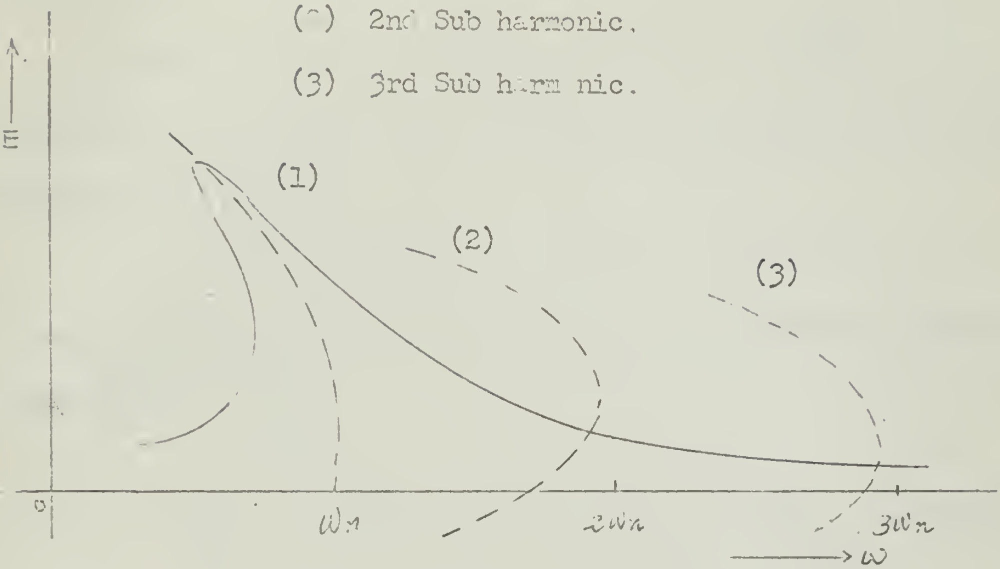
$$\begin{aligned} -E_{1B}\omega^2 - \alpha E_{1A}\,\omega + Ka_1E_{1B} + Ka_3\left(\frac{3}{4}E_{1A}^3 + \frac{3}{4}E_{1B}^3 + \frac{3}{2}E_{1/2}^2E_{1B}\right) & \\ = A\sin\theta - B\cos\theta & \end{aligned} \qquad (4\text{-}18)$$

From equation (4-18), it is the coefficient of the term of Cos $\frac{1}{2}\omega$ t, where $E_{1/2} \neq 0$, hence:

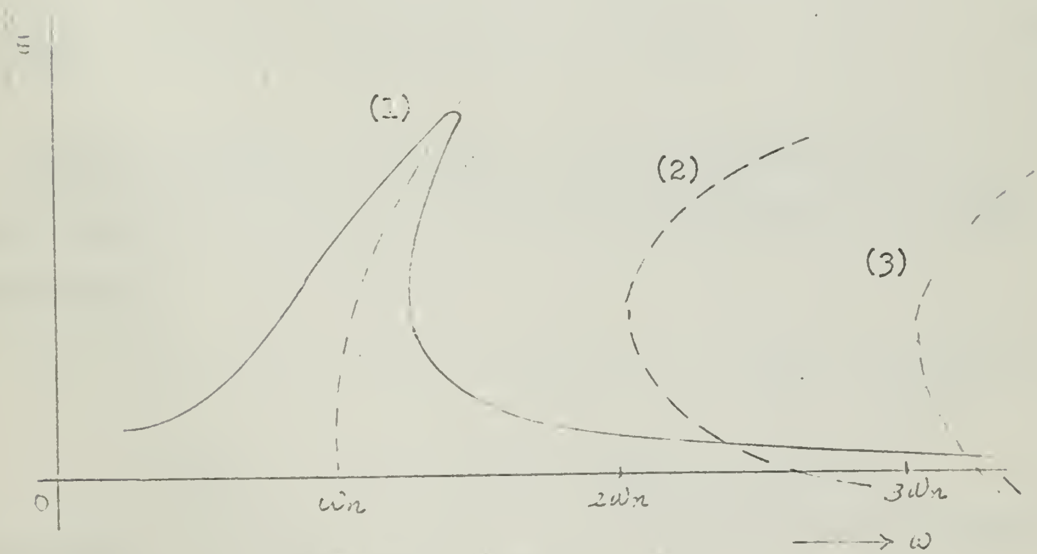
(1) Fundamental Frequency.

(2) 2nd Sub harmonic.

(3) 3rd Sub harmonic.



(a) Soft Saturation; $a_3 < 0$



(b) Hard Saturation; $a_3 > 0$

Fig. 4-4 Response Curves for Sub-harmonic Oscillations

$$\omega^2 - [\text{illegible}] = 0 \qquad (4\text{-}19)$$

Solve for $E_{1/2}$:

$$E_{1/2} = \pm\sqrt{\frac{1}{3Ka_3}\left([\text{illegible}] + 4Ka_1 + [\text{illegible}]\,E_{1A}^2 + [\text{illegible}]\,E_{1B}^2\right)} \qquad (4\text{-}20)$$

The value of $E_{1/2}$ should be real therefore, the existence of 2nd order sub-harmonic with damping:

$$\omega \le \sqrt{4Ka_1 + 6Ka_3E_{1A}^2 + 6Ka_3E_{1B}^2} \qquad a_3 \neq 0 \qquad (4\text{-}21)$$

From equations (4-16) to (4-18) for the first iteration method, for a_3 equal to zero: hence:

$$\omega^2 = 4Ka_1 \qquad (4\text{-}22)$$

$$(Ka_1 - \omega^2)E_{1A} + \alpha\omega\,E_{1B} = -\omega\,F\cos\theta - \alpha\omega\,F\sin\theta \qquad (4\text{-}23)$$

$$(Ka_1 - \omega^2)E_{1B} - \alpha\omega\,E_{1A} = \omega\,F\sin\theta - \alpha\omega\,F\cos\theta \qquad (4\text{-}24)$$

Solve E_{1A} and E_{1B} from equations of (4-22) to (4-24):

$$E_{1A} = \frac{4}{3}F\left[\left(1 - \frac{\alpha^2}{9\omega_n^2 + 4\alpha^2}\right)\cos\theta + \frac{\alpha}{\omega_n}\left(\frac{1}{2} - \frac{[\text{illegible}](3\omega_n^2 + \alpha^2)}{9\omega_n^2 + 4\alpha^2}\right)\right] \qquad (4\text{-}25)$$

$$E_{1B} = -\frac{[\text{illegible}]F\left(\alpha\omega_n\cos\theta + ([\text{illegible}] + 2\alpha^2)\sin\theta\right)}{9\omega_n^2 + 4\alpha^2} \qquad (4\text{-}26)$$

Inserting equations (4-25) and (4-26) into equation (4-21). A general equation for existence of 2nd order sub-harmonic oscillation with damping:

$$\omega \ge 2\left\{\omega_n^2 + \frac{5}{3}Ka_3F^2\left[\left(1 - \frac{\alpha^2}{9\omega_n^2 + 4\alpha^2}\right)\cos\theta + \frac{\alpha}{\omega_n}\left(\frac{1}{2} - \frac{[\text{illegible}](3\omega_n^2 + \alpha^2)}{9\omega_n^2 + 4\alpha^2}\right)\right]^2 \right.$$
$$\left. + 6Ka_3F^2\left[-\frac{[\text{illegible}]F\left([\text{illegible}]\cos\theta + [\text{illegible}]\sin\theta + [\text{illegible}]\sin\theta\right)}{9\omega_n^2 + 4\alpha^2}\right]^2\right\}^{1/2} \qquad (4\text{-}27)$$

For the case of θ equal to 0 or π, i.e., Sinθ equal to zero equation (4-27) becomes:

$$\omega \ge 2\sqrt{\omega_n^2 + \frac{5}{3}Ka_3F^2\left(1 - \frac{\alpha^2}{9\omega_n^2 + 4\alpha^2}\right)^2 + 6Ka_3\frac{[\text{illegible}]F^2[\text{illegible}]^2}{(9\omega_n^2 + 4\alpha^2)^2}} \qquad (4\text{-}28)$$

From equations (4-11) and (4-28) they were shown that either the system with damping or without it the second order sub-harmonic oscillation only existed with a frequency less than the frequency of two times the natural frequency of linear system for the case of a_3 less than zero, and greater than this frequency for the case of a_3 greater than zero.

4-4 Sub-harmonic Oscillations of Order 3 without Damping:

Typical waveforms with jump phenomena of 3rd order sub-harmonic forced oscillation is shown in Fig. 4-5; (a) shows the transient from the 2nd order to the 3rd order sub-harmonic; (b) is shown reversing.

For investigating the existence conditions of 3rd sub-harmonic forced oscillation recall the basic equation of (3-11); and leads:

$$\ddot{E} + Ka_1E + Ka_3E^3 = -A\,\mathrm{Cos}(\omega t + \theta) \qquad (3\text{-}12)$$

Assume one solution of equation (3-12) is:

$$E = E_{1/3}\,\mathrm{Cos}\,\frac{1}{3}\omega t + E_1\,\mathrm{Cos}\,\omega t \qquad (4\text{-}29)$$

in which $E_{1/3}$ is the amplitude of 3rd order sub-harmonic oscillation, hence:

$$\ddot{E} = -\frac{\omega^2}{9}E_{1/3}\,\mathrm{Cos}\,\frac{1}{3}\omega t - \omega^2E_1\,\mathrm{Cos}\,\omega t \qquad (4\text{-}30)$$

$$E^3 = \frac{3}{4}E_{1/3}(E^2_{1/3} + E_{1/3}E_1 + 2E_1^2)\,\mathrm{Cos}\frac{1}{3}\omega t$$

$$\frac{1}{4}(E_{1/3}^3 + 6E_{1/3}^2E_1 + 3E_1^3)\,\mathrm{Cos}\,\omega t + \ldots\ldots \qquad (4\text{-}31)$$

From equation (4-29) to (4-31) and equation (3-12) leads:

$$(Ka_1 - \frac{\omega^2}{9})E_{1/3} + \frac{3}{4}Ka_3E_{1/3}(E_{1/3}^2 + E_{1/3}E_1 + 2E_1^2) = 0 \qquad (4\text{-}32)$$

$$(Ka_1 - \omega^2)E_1 + \frac{1}{4}Ka_3(E_{1/3}^3 + 6E_{1/3}^2E_1 + 3E_1^3) = -A \qquad (4\text{-}33)$$

Equation (4-32) $E_{1/3}$ is not equal to zero, hence:

$$E_{1/3}^2 + E_1E_{1/3} + 2E_1^2 + \frac{4}{3Ka_3}(Ka_1 - \frac{\omega^2}{9}) = 0 \qquad (4\text{-}34)$$

Solve $E_{1/3}$ from equation (3-34).

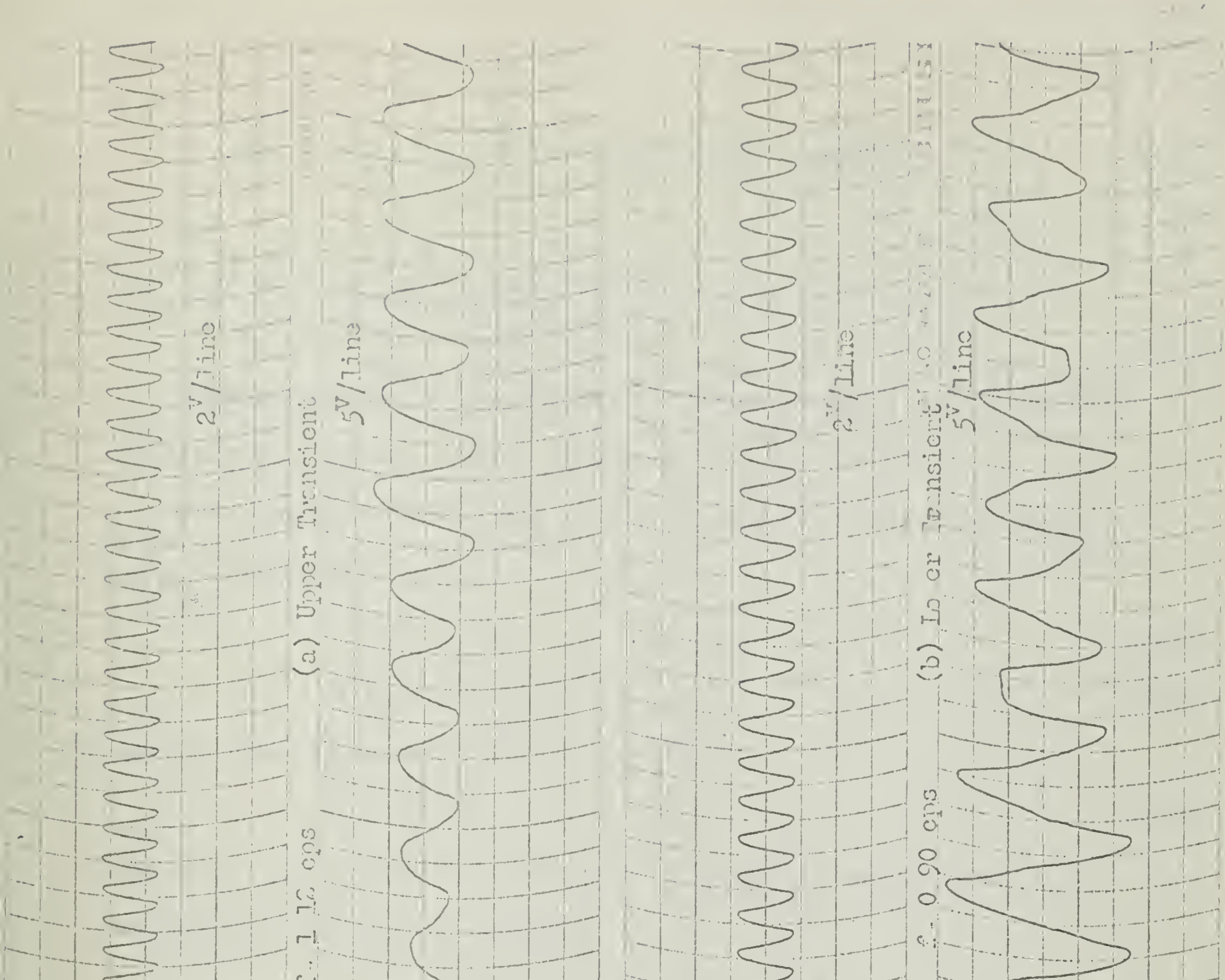


Fig.4-5 Waveforms of Upper and Lower Jump Transient
for A Typical 3rd Order Sub-harmonic
Forced Oscillation

$$E_{1/3} = [illegible] \qquad (4\text{-}35)$$

The value of $E_{1/3}$ must be real, then:

$$\omega^2 \geq 9Ka_1 + \frac{27Ka_3}{16} \times 7E_1^2 \qquad a_3 \gtrless 0 \qquad (4\text{-}36)$$

Equation (4-36) is the condition for the existenceof sub-harmonic oscillation of order 3 of a without damping system:

From equations (4-32) and (4-33), by the first iteration method, for $a_3 = 0$; then:

$$\omega^2 = 9\ Ka_1 \qquad (4\text{-}37)$$

$$(Ka_1 - \omega^2)E_1 = -\omega^2 F \qquad (4\text{-}38)$$

Solve E_1 from equation (4-37) and (4-38):

$$E_1 = \frac{9}{8}\ F \qquad (4\text{-}39)$$

Inserting equation (4-39) in equation (4-36) the condition for a 3rd sub-harmonic oscillation becomes:

$$\omega \lessgtr 3\sqrt{\omega_1^2 + \frac{21}{16} \times \frac{81}{64} Ka_3 F^2} \qquad (4\text{-}40)$$

Also the response equation of equation 4-34 becomes:

$$E_{1/3}^2 + \frac{1}{3}E_{1/3}F + \frac{81}{[illegible]}F^2 + [illegible] = 0 \qquad (4\text{-}41)$$

The equation of (4-41) represents an ellipse or a hyperbola in the $E_{1/3}$ vs ω plane depending on the sign of a_3, also ω has a maximum when $a_3 < 0$, and a minimum when $a_3 > 0$. A sketch of a 3rd sub-harmonic forced oscillation is shown in Fig. 4-4.

4-5 Sub-harmonic Oscillation of Order 3 with Damping:

Recall equation (3-41):

$$\ddot{E} + \alpha\dot{E} + Ka_1E + Ka_3E^3 = -ACos(\omega t + \theta) - BSin(\omega t + \theta) \qquad (3\text{-}41)$$

Assume one solution of equation (3-41) is:

$$E = E_{1/3}Cos\,\frac{1}{3}\omega t + E_{1A}Cos\,\omega t + E_{1B}Sin\,\omega t \qquad (4\text{-}42)$$

Since:

$$\dot{E} = E_{1/3}\frac{\omega}{3}\,\mathrm{Sin}\,\frac{1}{3}\omega t - E_{1A}\,\omega\,\mathrm{Sin}\,\omega t + E_{1B}\,\omega\,\mathrm{Cos}\,\omega t \qquad (4\text{-}43)$$

$$\ddot{E} = -\frac{\omega^2}{9}E_{1/3}\,\mathrm{Cos}\,\frac{1}{3}\omega t - E_{1A}\,\omega^2\,\mathrm{Cos}\,\omega t - E_{1B}\,\omega^2\,\mathrm{Sin}\,\omega t \qquad (4\text{-}44)$$

$$E^3 = \frac{3}{4}E_{1/3}(E_{1/3}{}^2 + E_{1/3}E_{1A} + 2E_{1A}{}^2 + 2E_{1B}{}^2)\mathrm{Cos}\,\frac{1}{3}\omega t +$$

$$\frac{3}{4}E_{1/3}{}^2E_{1B}\mathrm{Sin}\frac{1}{3}\omega t + 1/4(E_{1/3}{}^3 + 6E_{1/3}{}^2E_{1A} + 3E_{1A}{}^3 +$$

$$3E_{1A}E_{1B}{}^2)\ \mathrm{Cos}\,\omega t + \frac{3}{4}E_{1B}(E_{1B}{}^2 + 2E_{1/3}{}^2 + E_{1A}{}^2)\ \mathrm{Sin}\,\omega t + \ldots \qquad (4\text{-}45)$$

From equation (4-42) to (4-44) and (3-41), by equaling the coefficients of $\mathrm{Cos}\,\frac{1}{3}\omega t$, $\mathrm{Sin}\,\omega t$, $\mathrm{Cos}\,\omega t$, leads:

[illegible] (4-46)

[illegible] (4-47)

[illegible] (4-48)

Rearrange equation (4-46) and solve it for $E_{1/3}$ in terms of , E_{1A}, and E_{1B}:

[illegible] (4-49)

here $E_{1/3}$ should be real, hence:

[illegible] (4-50)

or:

[illegible] (4-51)

Equation (4-51) is the condition of 3rd sub-harmonic oscillation existing in saturation nonlinearity with damping system.

By the iteration method, equation (4-46) to (4-48) leads:

$$\omega^2 = 9Ka_1 \qquad (4\text{-}53)$$

$$(Ka_1 - \omega^2)E_{1A} + \alpha\omega E_{1B} = -\omega^2 F\mathrm{Cos}\,\theta - \omega F\mathrm{Sin}\theta \qquad (4\text{-}54)$$

$$(Ka_1 - \omega^2)E_{1B} - \alpha\omega E_{1A} = \omega^2 F\mathrm{Sin}\,\theta - \omega F\mathrm{Cos}\theta \qquad (4\text{-}55)$$

Solve equation (4-53) to (4-55),

$$E_{1A} = \text{[illegible]} \tag{4-56}$$

$$E_{1B} = \text{[illegible]} \tag{4-57}$$

Substituting equations (4-56) (4-57) into equation (4-52) and becomes in a general form:

$$\omega \geq \text{[illegible]} \tag{4-58}$$

In the case of θ equal to 0 or π, i.e. $\sin\theta$ equal to zero equation (4-58) becomes:

$$\omega \geq 3\sqrt{\text{[illegible]}} \tag{4-59}$$

In conclusion, the condition for existence of sub-harmonic oscillation in a nonlinear feedback control system depends on the characterics of the nonlinear element and the amplitude of input. The response curve of any order sub-harmonic oscillation with a saturation nonlinearity is either an ellipse or a hyperbola which depends on the characteristic of nonlinearity. Also the frequency of a sub-harmonic oscillation existed may be either a maximum when a soft saturation nonlinearity is presented or a minimum when a hard saturation nonlinearity is presented. The limit value of minimum or maximum is almost equal to the number of sub-harmonic order times the natural frequency of linear system.

4-6 Analog Computer Analysis for a 2nd Order Sub-harmonic Oscillation System:

Consider a system with a block diagram shown in Fig. 4-6:

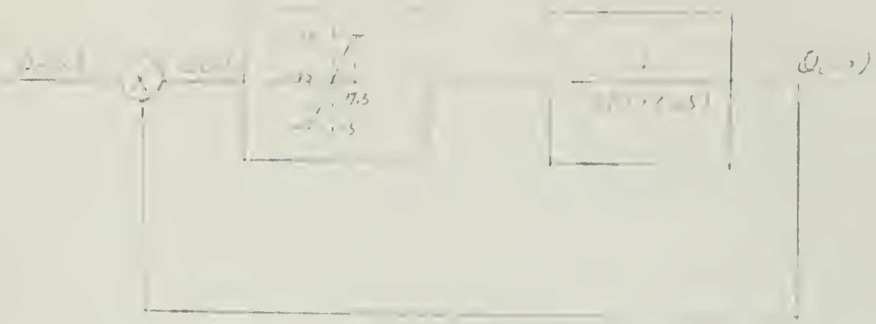


Fig. 4-6 Block Diagram of a 2nd Sub-harmonic Oscillation System.

Assume the restoring force function of nonlinearity is:

$$f(E) = 2E - 0.0048E^3 \tag{4-59}$$

The characteristic curve for equation (4-59) is shown in Fig. 4-7, in which:

E = 1.00; 2.50; 5.00; 7.50; 10.00; 12.50; 15.00.

f(E) = 1.995; 4.295; 9.400; 12.97; 15.20; 15.40; 14.80.

For the simulation of restoring force function of nonlinearity from computer, it is approximated from two straight lines, one of which with a slope of 2; and the other with a slope of zero. The circuit setup in the computer is identical with the circuit shown in Fig. 3-15. The only difference is the value of bias voltage to be used. The characteristic curve of restoring force function from the computer is shown in Fig. 4-8.

An operational diagram setup in the analog computer is almost the same with the diagram of Fig. 3-20, only one difference is that the value of constant to be used.

The results from the computer are condensed in Table 4-1 and Fig. 4-9. Only the response curve is of interest, hence, only one response curve in the error vs frequency plane is presented. It is shown that there are two

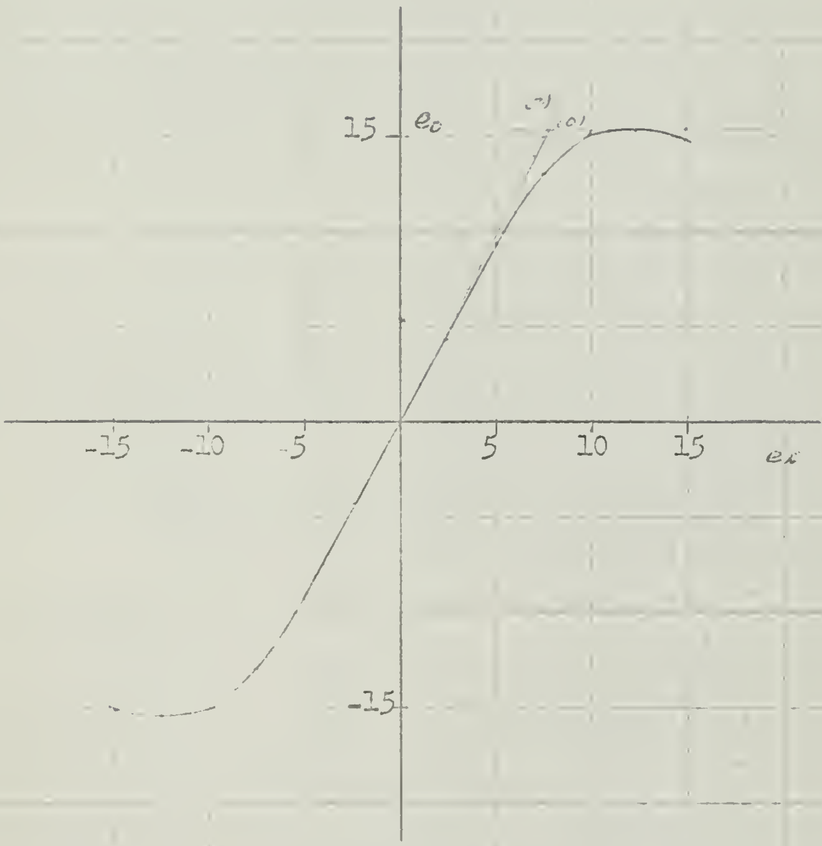


Fig.4.7 Characteristic Curve Plotted of: $f(E) = 2E - 0.0048E^3$

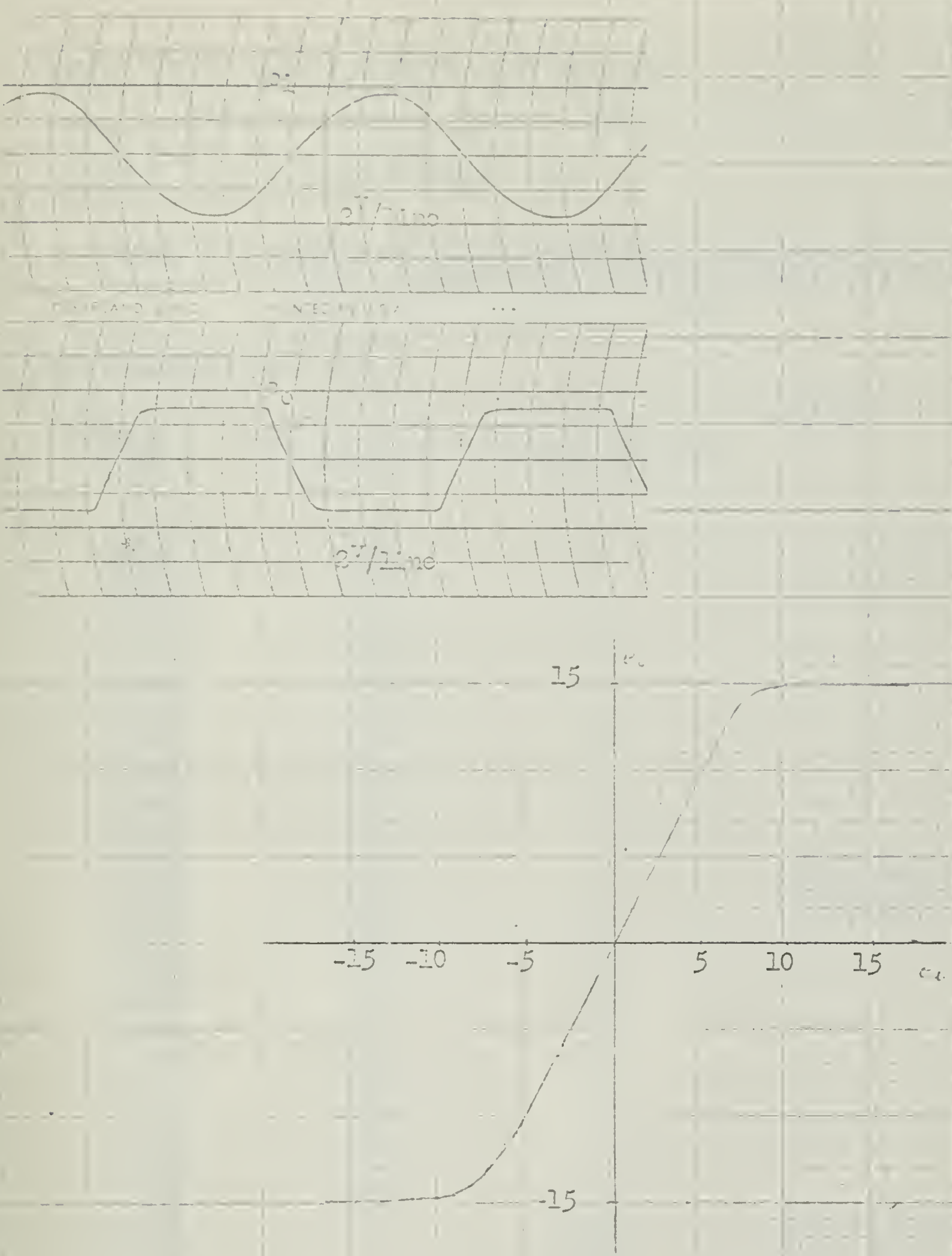


Fig.4 8 Computer Characteristic Curve of: $f(E) = 2E - 0.0048E^3$

jumps, one is the fundamental frequency jump, and the other is the second order sub-harmonic frequency jump. These jump phenomenas are shown schematically in the diagram of Fig. 4-9.

Table 4-1: Data for a 2nd Order Sub-harmonic Oscillation Response Curve, in which: $f(E) = 2E - 0.0048E^3$; $K = 4, \alpha = 0.25$; and $\theta_r(t) = 2.5 \, Cos(\omega t + \theta)$

(a) Frequency Increasing:

Frequency of Input (ω)	Amplitude of 1st Harmonic (E_1)	Amplitude of 2nd Sub-harmonic (E_2)	Remarks
0.628	0.22	/	
0.754	0.33	/	
0.880	0.45	/	
1.000	0.50	/	
1.130	0.63	/	
1.260	1.00	/	
1.380	1.15	/	
1.510	1.50	/	
1.640	2.00	/	
1.760	2.50	/	
1.890	3.00	/	
1.950	3.25	/	
1.980*	10.50	/	*1st harmonic upper jump
2.010	10.00	/	
2.200	9.00	/	
2.510	7.30	/	
2.810	6.00	/	
2.950	/	/	
3.140	5.00	/	
3.460	4.50	/	
3.770	4.00	/	
4.090	3.75	/	
4.400	3.50	/	
4.710	3.25	/	
5.020	/	4.75*	*2nd sub-harmonic upper jump
5.150	/	4.50	
5.270	/	3.50	
5.340	2.50	/	

(b) Frequency Decreasing:

Frequency of Input (ω)	Amplitude of 1st Harmonic ([illegible])	Amplitude of 2nd Sub-harmonic ([illegible])	Remarks
5.340	2.50	/	
5.270	/	3.50	
5.150	/	4.50	
5.020	/	4.75	
4.710	/	5.50	
4.400	/	6.50	
4.090	/	7.80	
3.770	/	9.50	
3.460	/	11.00	
3.140	/	13.50	
2.950	/	14.50*	*2nd sub-harmonic lower jump
2.810	6.00	/	
2.510	7.25	/	
2.200	8.75	/	
2.010	10.00	/	
1.890	11.00	/	
1.760	12.25	/	
1.640	13.00	/	
1.510	15.00	/	
1.380	17.00	/	
1.260	21.20	/	
1.130	24.00	/	
1.000	25.50	/	
0.880	0.45*	/	*1st Harmonic lower jump
0.754	0.33	/	
0.628	0.22	/	

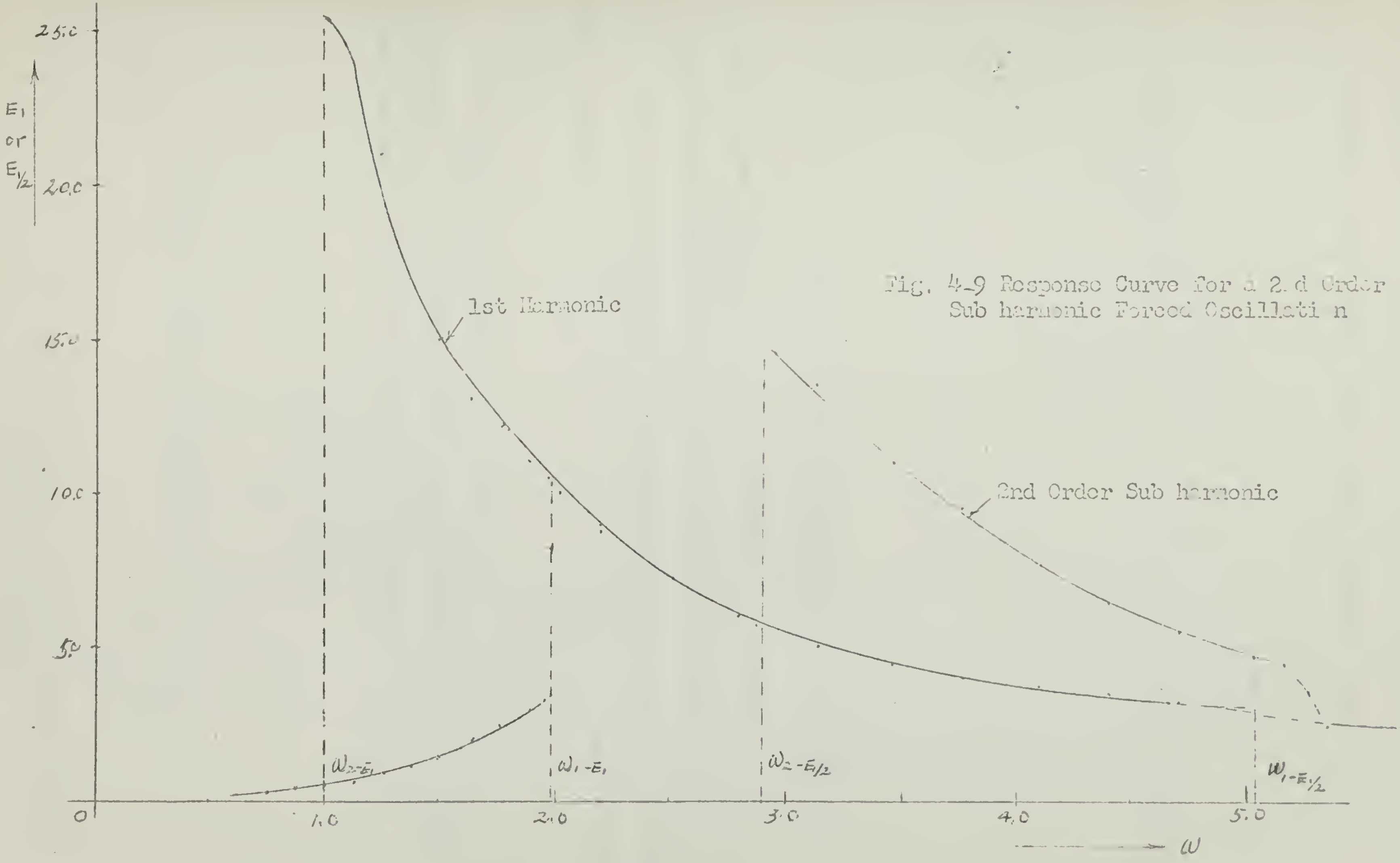


Fig. 4-9 Response Curve for a 2nd Order Sub harmonic Forced Oscillation

4-7 Analog Computer Analysis for a 3rd Order Sub-harmonic Oscillation System:

Consider a system with a block diagram shown in Fig. 4-10:

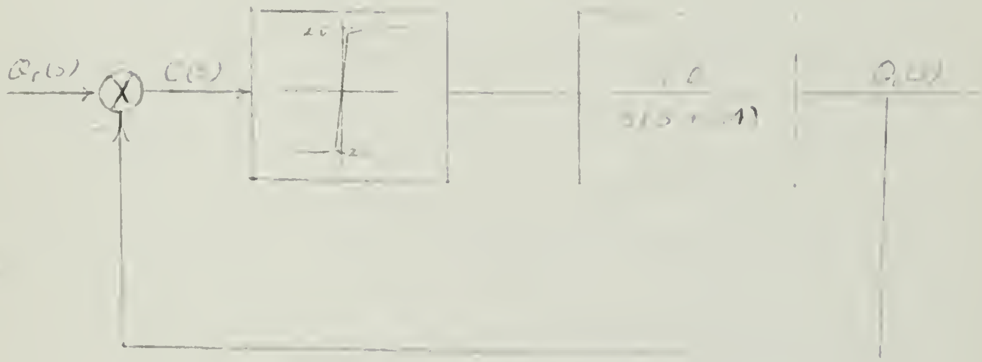


Fig. 4-10 Block Diagram of a 3rd Sub-harmonic Oscillation System

in which assume the nonlinearity of system is an "ON-OFF" element, a sketch of the characteristic for the computer simulation is shown in Fig. 4-11, it is approximated from three straight lines; one of which with a slope of 15.0; one of which with a slope of 0.1; and the other is zero:

The circuit setting up in the computer is shown in Fig. 4-12, all components to be used are schematically indicated with the diagram. The characteristic curve from the computer is shown in Fig. 4-13.

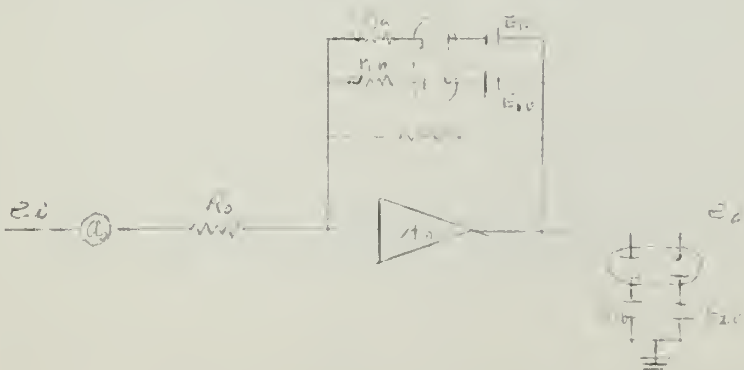


Fig. 4-12 Function Generator Diagram for an "ON-OFF" Nonlinearity

Fig. 4 ii Characteristic Sketch of An "On-Off" Nonlinear Element

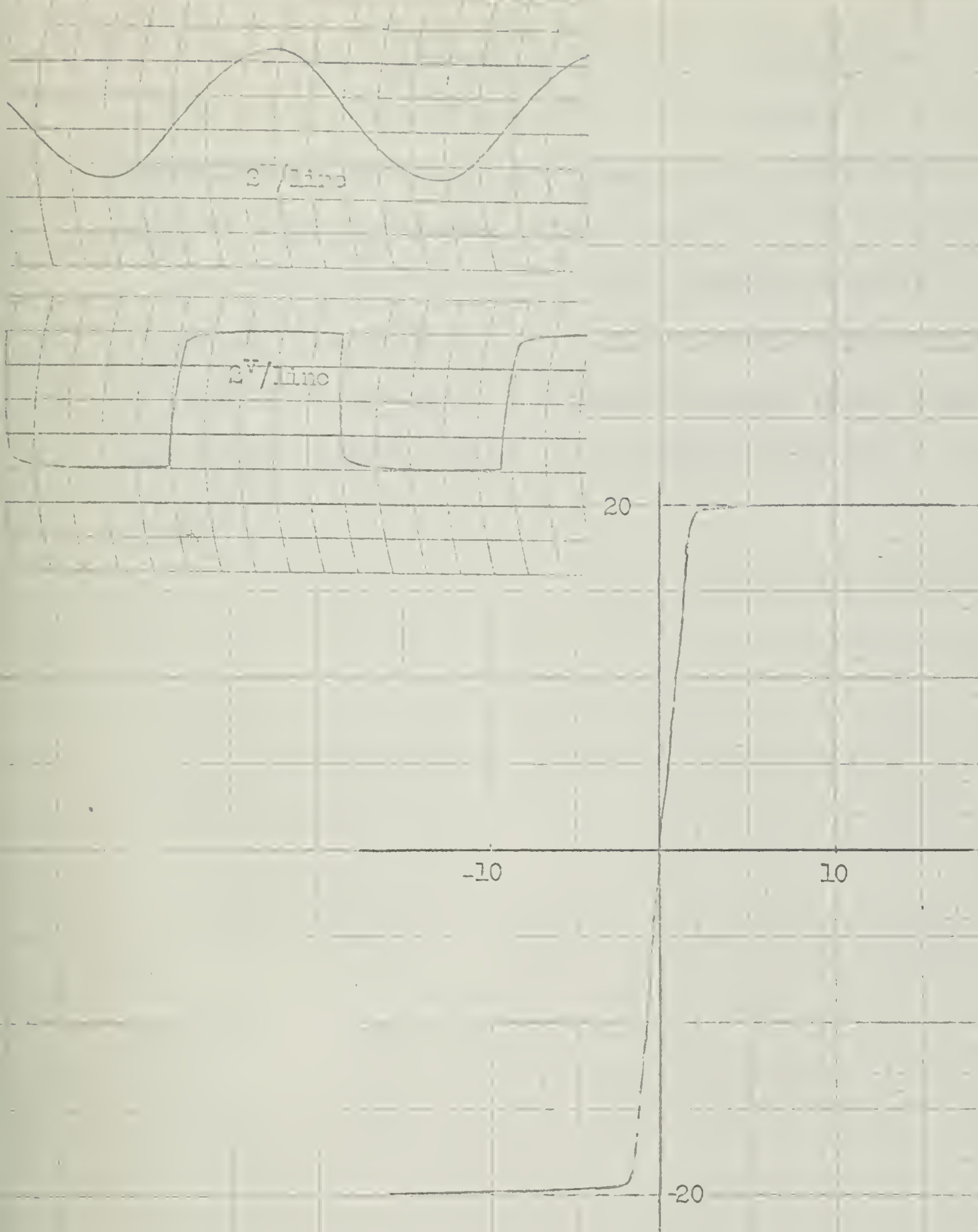


Fig.4-13 Characteristic Curve of An "On-Off" Nonlinearity From The Analog Computer

An operational diagram for setting up the analog computer is shown in Fig. 4-14, in which the value of all components to be used are schematically indicated in the diagram. The scaling factor for the computer setup is the same as diagram of Fig. 3-20 to be used.

The results from the computer are condensed in Table 4-2, and with a response curve shown in Fig. 4-15. It is shown that, there is a pair of jumps for each harmonic oscillation response curve, these jump phenomenas within frequency range is accompanied shown with the diagram of Fig. 4-15.

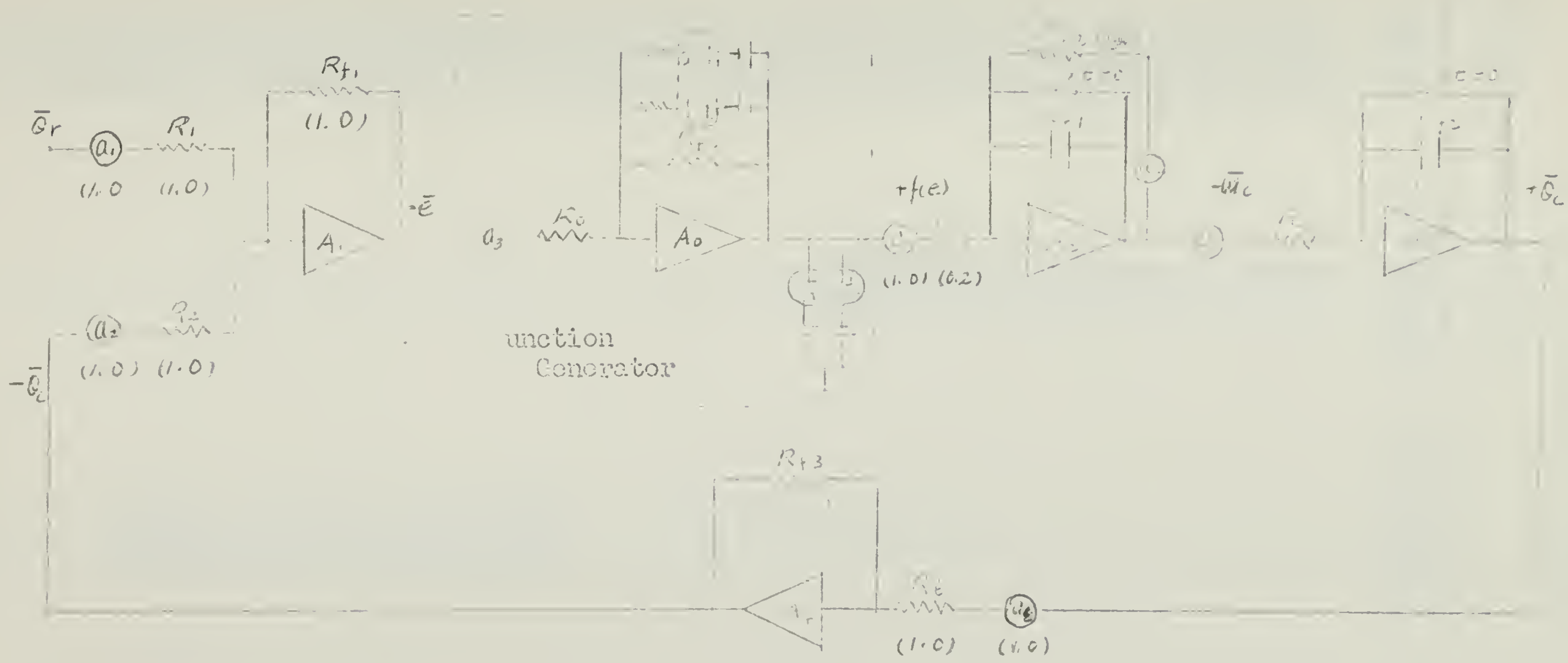


Fig.4 14 Operational Diagram of Analog Computer Setting up for System of Block Diagram in Fig.4.10.

Table 4-2: Data for a 3rd Order Sub-harmonic Oscillation Response Curve from an "ON-OFF" Nonlinearity System and; $\zeta = 0.4$; $K = 10.0$; $\theta_r(t) = 8.0 \cos(\omega t + \theta)$

(a) Frequency Increasing:

Frequency of Input (ω)	Amplitude of 1st Harmonic (E_1)	Amplitdue of 2nd Sub-har. ($E_{1/2}$)	Amplitude of 3rd Sub-har. ($E_{1/3}$)	Remarks
0.628	0.60	/	/	
0.880	1.25	/	/	
1.130	2.25	/	/	
1.380	3.50	/	/	
1.640	4.75	/	/	
1.760	5.75	/	/	
1.890	7.50	/	/	
2.010*	50.00	/	/	*1st Harmonic
2.140	46.00	/	/	Upper Jump
2.270	42.50	/	/	
2.390	39.00	/	/	
2.510	35.00	/	/	
2.630	31.50	/	/	
2.760	28.00	/	/	
2.890	26.00	/	/	
3.010	22.50	/	/	
3.140	20.00	/	/	
3.260	/*	/	/	*Transient and
4.270	/*	/	/	unstable
4.390°	/	/	/	°Transient from
4.520	/	/	/	1st Harmonic
4.650	/	/	/	to 2nd Sub-har.
4.770	/	/	/	
4.990°	/	/	/	
5.020*	/	/	/	
5.140	/	20.00	/	*2nd Sub-har.
5.270	/	18.00	/	Upper Jump
5.400	/	16.70	/	
5.520	/	16.00	/	
5.650	/	15.00	/	
5.770	/	13.80	/	
5.900	/	13.50	/	
6.030	/	12.70	/	
6.150	/	12.30	/	
6.280	/	11.90	/	
6.400	/	11.30	/	
6.530	/	11.00	/	
6.650	/	10.60	/	
6.780	/	10.20	/	
6.910*	/	10.00	/	*3rd Sub-har.
7.030	/	/	35.00	Upper Jump
7.160	/	/	32.50	
7.280	/	/	28.50	
7.410	/	/	27.20	
7.530	/	/	26.00	

(b) Frequency Decreasing:

Frequency of Input (ω)	Amplitude of 1st Harmonic (E_1)	Amplitude of 2nd Sub-har. ($E_{1/2}$)	Amplitude of 3rd Sub-har. ($E_{1/3}$)	Remarks
7.530	/	/	26.00	
7.410	/	/	27.20	
7.280	/	/	28.50	
7.160	/	/	30.50	
7.030	/	/	32.50	
6.910	/	/	35.00	
6.780	/	/	36.50	
6.650	/	/	37.50	
6.530	/	/	40.00	
6.400	/	/	42.50	
6.280	/	/	43.75	
6.150	/	/	47.00	
6.030	/	/	50.00	
5.900	/	/	52.50	
5.770	/	/	56.00	
5.650*	/	13.75	/	*3rd Sub-har.
5.520	/	15.00	/	Lower Jump
5.400	/	16.00	/	
5.270	/	16.70	/	
5.140	/	17.50	/	
5.020	/	19.50	/	
4.900	/	21.50	/	
4.770	/	23.00	/	
4.650	/	25.00	/	
4.520	/	32.00	/	
4.390	/	36.00	/	
4.270 °*	/	/	/	°*2nd Sub-har.
4.140	/	/	/	Lower Jump
3.260°	/	/	/	and Transient
3.140	20.00	/	/	Unstable Region
3.010	22.50	/	/	
2.890	26.00	/	/	
2.760	27.50	/	/	
2.630	31.00	/	/	
2.510	35.00	/	/	
2.390	39.00	/	/	
2.270	42.00	/	/	
2.140	45.50	/	/	
2.010	51.00	/	/	
1.890	55.00	/	/	
1.760	52.50	/	/	
1.640*	5.00	/	/	*1st Harmonic
1.510	4.25	/	/	Lower Jump
1.380	3.50	/	/	
1.130	2.30	/	/	
0.880	1.25	/	/	
0.628	0.60	/	/	

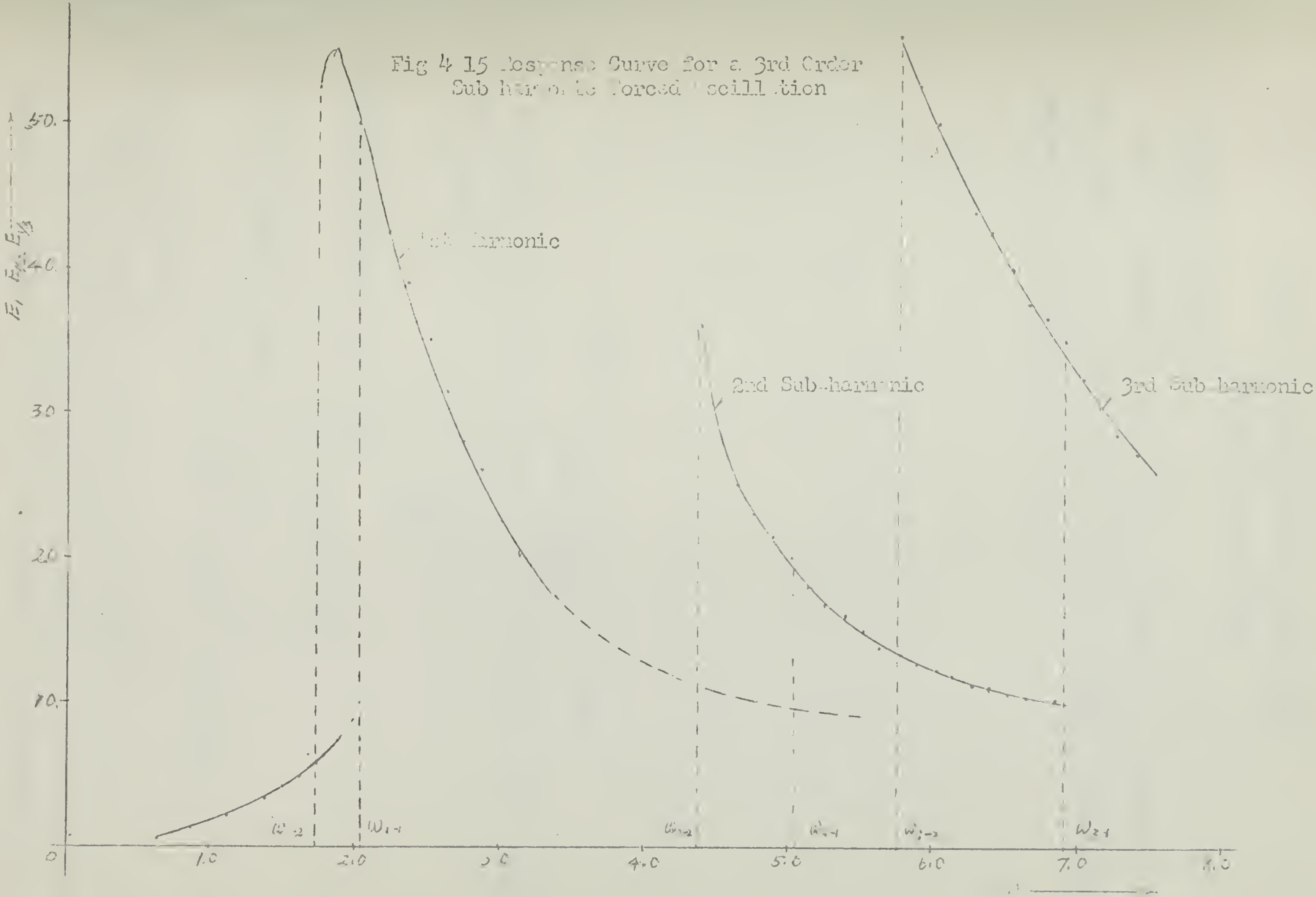


Fig 4.15 Response Curve for a 3rd Order Subharmonic Forced Oscillation

CHAPTER V

FORCED OSCILLATION FOR THE HIGHER ORDER FEEDBACK CONTROL SYSTEM

5-1 General Description:

Up to the Chapter 4, only forced oscillations in the case of the 2nd order system are discussed. In the extension to higher order systems, consider a block diagram shown in Fig. 5-1:

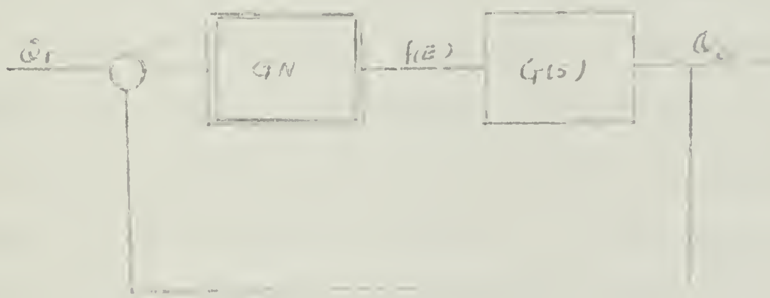


Fig. 5-1 Block Diagram of a Feedback Control System

in which G(s) is an equivalent transfer function of the linear element of the system, it can be written in the form:

$$G(s) = K \frac{Q(s)}{P(s)} \tag{5-1}$$

and with a result for the system:

$$P(s)\ E + Q(s)\ \ Kf(E) = P(s) \tag{5-2}$$

Where P(s) may be any order of s; as:

$$P(s) = s^{n} + b_1 s^{n-1} + b_2 s^{n-2} + \dots\dots + b_n \tag{5-3}$$

Similarily Q(s) may be

$$Q(s) = s^{m} + c_1 s^{m-1} + c_2 s^{m-2} + \dots\dots\dots\dots + c_m \tag{5-4}$$

In general the frequency responsive element G(s) following the non-linearity is of such a nature as to attenuate the high frequency, that means the order of P(s) should be higher than the order of Q(s). The order of system is n. Inserting equations (5-3) and (5-4) into (5-2) and:

$$\left(\frac{d^n}{dt^n} + a_1\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_n\right) = \left(\frac{d^m}{dt^m} + c_1\frac{d^{m-1}}{dt^{m-1}} + \cdots + c_m\right)$$

$$\left(\frac{d^n}{dt^n} + b_1\frac{d^{n-1}}{dt^{n-1}} + \cdots + b_n\right) \qquad (5\text{-}5)$$

Equation (5-5) is a general differential equation form of the forced oscillation of any order feedback control system.

5-2 Harmonic Linearization and Iteration Method for Investigating the Equivalent Gain of a Nonlinearity:

After a restoring force function of the nonlinear element has been defined, the equivalent complex gain, $N(E_1)$ of the nonlinear element is determined by the ratio of the complex amplitude of the first harmonic of the output to the amplitude of the harmonically varying input quantity.

Recall the block diagram of Fig. 5-1, if the system is a 2nd order system and with a restoring forcing function; $f(E) = a_1E + a_3E^3$, and:

$$\frac{K\,N(E)\,E}{s\,(s+\alpha)} = \theta_c \qquad (5\text{-}6)$$

or:

$$\ddot{E} + \alpha\dot{E} + K\,N(E)\,E = \ddot{\theta}_r + \alpha\dot{\theta}_r \qquad (5\text{-}7)$$

Assume the input function is

$$\theta_r = F\,\mathrm{Cos}(\omega t + \theta) \qquad (5\text{-}8)$$

and one solution of E:

$$E = E_1\,\mathrm{Cos}\,\omega t \qquad (5\text{-}9)$$

and with a result:

$$(-\omega^2 E_1 + K\,N(E_1)E_1)\mathrm{Cos}\,\omega t - \alpha\omega E_1\mathrm{Sin}\,\omega t = -A\,\mathrm{Cos}(\omega t + \theta)$$

$$-\,B\mathrm{Sin}(\omega t + \theta) \qquad (5\text{-}10)$$

Comparing the result with the equation of (3-42) the equivalent gain:

$$N(E_1) = a_1 + \frac{3}{4}\,a_3E_1^{\,2} \qquad (5\text{-}11)$$

For a general form expression; the equivalent gain of a nonlinearity in terms of the amplitude of input is developed by Ya. Z. Tsypkin. The result for a symmetrical nonlinearity with input of equation (5-9) is:

$$N(E_1) = \frac{2}{3E_1}\left[f(E_1) + f(E_{1/2})\right] \tag{5-12}$$

If more accuracy is required, the equivalent gain $N(E_1)$ may be:

$$N(E_1) = \frac{1}{3E_1}\left[f(E_1) + f(E_1/2) + \sqrt{3}\, f(\sqrt{3}\, E_1/2)\right] \tag{5-13}$$

For the case of a nonlinearity with a restoring force function, $f(E) = a_1E + a_3E^3$, the results from equations (5-12) and (5-13) are exactly the same as equation (5-11). Hence, for any nonlinearity the case of first harmonic input the equivalent gain can be expressed either in the form of equation (5-12) or (5-13).

Equation (5-12) or (5-13) not only substantially simplifies the calculation for a second order forced oscillation, it is also probably a most important for investigating self oscillation or forced oscillations in a higher order system with single or multiple nonlinearities in a feedback control system.

5-3 Forced Oscillations for a Higher Order Feedback Control System:

Consider a system with a linear transfer function:

$$G(s) = \frac{K(s + c_1)}{s(s + p_1)(s + p_2)(s + p_3)} \tag{5-14}$$

or in the form:

$$G(s) = \frac{K(s + c)}{s^4 + b_1s^3 + b_2s^2 + b_3s} \tag{5-15}$$

From the equation (5-5), the differential equation for the system of equation (5-15):

$$\left(\frac{d^4}{dt^4} + b_1\frac{d^3}{dt^3} + b_2 \frac{d^2}{dt^2} + b_3 \frac{d}{dt} \right)E + K\left(\frac{d}{dt} + c\right)f(E)$$

$$= \left(\frac{d^4}{dt^4} + b_1 \frac{d^3}{dt^3} + b_2 \frac{d^2}{dt^2} + b_3 \frac{d}{dt}\right) \theta_r \tag{5-16}$$

in which;

$$E = E_1 \mathrm{Cos}\,\omega t$$

$$[illegible] = F\mathrm{Cos}(\omega t + \theta)$$

and;

$$f(E) = a_1 E_1 \mathrm{Cos}\,\omega t + \frac{1}{4} a_3 E_1^3 (3\mathrm{Cos}\,\omega t + \mathrm{Cos}\,3\omega t)$$

Inserting these equations into equation (5-16) and with a result:

$$(E_1\omega^4 - b_2E_1\omega^2 + Ka_1CE_1 + \frac{3}{4}Ka_3CE_1^3)\cos\omega t$$

$$+ (b_1E_1\omega^3 - b_3E_1\omega - Ka_1E_1\omega - \frac{3}{4}Ka_3E_1^3\omega)\sin\omega t$$

$$+ \frac{1}{4}Ka_3CE_1^3\cos 3\omega t - \frac{3}{4}Ka_3E_1^3\omega\sin 3\omega t$$

$$= F\omega[\omega(\omega^2 - b_2)\cos\theta + (b_1\omega^2 - b_3)\sin\theta]\cos\omega t$$

$$- F\omega[\omega(\omega^2 - b_2)\sin\theta - (b_1\omega^2 - b_3)\cos\theta]\sin\omega t \qquad (5\text{-}17)$$

Neglect the higher order harmonic terms and equate the coefficients of $\mathrm{Cos}\,\omega t$ and $\mathrm{Sin}\,\omega t$:

$$E_1(\omega^4 - b_2\omega^2 + Ka_1C + \frac{3}{4}Ka_3CE_1^2)$$

$$= F\omega[(\omega^2 - b_2)\omega\cos\theta + (b_1\omega^2 - b_3)\sin\theta] \qquad (5\text{-}18)$$

$$E_1[(b_1\omega^2 - b_3) - (Ka_1 + \frac{3}{4}Ka_3E_1^2)]\omega$$

$$= F\omega[(\omega^2 - b_2)\omega\sin\theta - (b_1\omega^2 - b_3)\cos\theta] \qquad (5\text{-}19)$$

Squaring equations (5-18) and (5-19) and adding:

$$E_1^2(\omega^4 - b_2\omega^2 + Ka_1C + \frac{3}{4}Ka_3CE_1^2)^2 + E_1^2\omega^2[(b_1\omega^2 - b_3) - (Ka_1 + \frac{3}{4}Ka_3E_1^2)]^2$$

$$= F^2\omega^2[\omega^2(\omega^2 - b_2)^2 + (b_1\omega^2 - b_3)^2] \qquad (5\text{-}20)$$

Equation (5-20) is the equation of response curve of the system of equation (5-15). If the constants of system are know, a response curve in the $E_1 \sim \omega$ plane can be plotted.

For the investigation of conditions for existence of forced oscillation rearrange the equation (5-20) and put:

$$X = a_1 + \frac{3}{4} a_3 E^2$$

Hence:

$$[illegible] \qquad (5\text{-}21)$$

or;

$$[illegible] \qquad (5\text{-}22)$$

The condition for existence of forced oscillation is found from equation (5-22) by means that the solution of ω should be real. It is to be noted that the value of X is the equivalent gain of nonlinearity, its value should be larger than zero, as it was proved in Chapter 3.

For the phase investigating recall the equation (5-19) and re-write:

$$[illegible] \qquad (5\text{-}23)$$

or:

$$\sin(\theta - \phi) = [illegible] \qquad (5\text{-}24)$$

hence:

$$\theta = \sin^{-1} [illegible] + \phi \qquad (5\text{-}25)$$

in which the value of;

$$\phi = \tan^{-1}\{(b_1\omega^2 - b_0)/\omega(\omega^2 - b_2)\} \qquad (5\text{-}26)$$

Equation (5-25) is shown that the value of θ is a function of ω and the amplitude of first harmonic of error signal, if the system constants is fixed.

If we use the equivalent gain for the system investigation, and with a result:

$$\theta_c = \frac{G(j\omega)N(E_1)}{1 + N(E_1)G(j\omega)}\theta_r \tag{5-27}$$

or:

$$\frac{\theta_c}{\theta_r} = \frac{G(j\omega)K(a_1 + \frac{3}{4}a_3E_1^2)}{1 + G(j\omega)K(a_1 + \frac{3}{4}a_3E_1^2)} \tag{5-28}$$

Where E_1 is the amplitude of first harmonic of error signal of forced oscillation.

Equation (5-28) is similar to a linear system, the techniques for linear system are applicable.

5-4. Analog Computer Analysis for a Higher Order System:

Consider a feedback control system with a transfer function and a nonlinearity are shown in Fig. 5-2:

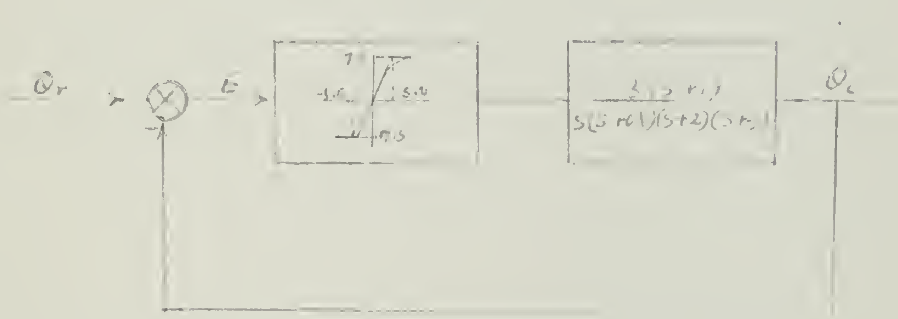


Fig. 5-2 Block Diagram for a Typical Feedback Control System

in which the restoring force function is:

$$f(E) = 2E + 0.020E^3 \tag{5-29}$$

The characteristic curve of restoring force function of nonlinearity from the calculation is shown in Fig. 5-3. Where:

E = 1.00; 2.00; 3.00; 4.00; 5.00; 6.00; 7.00; 7.50;

f(E) = 1.98; 3.84; 5.46; 6.72; 7.50; 7.68; 7.15; 6.60:

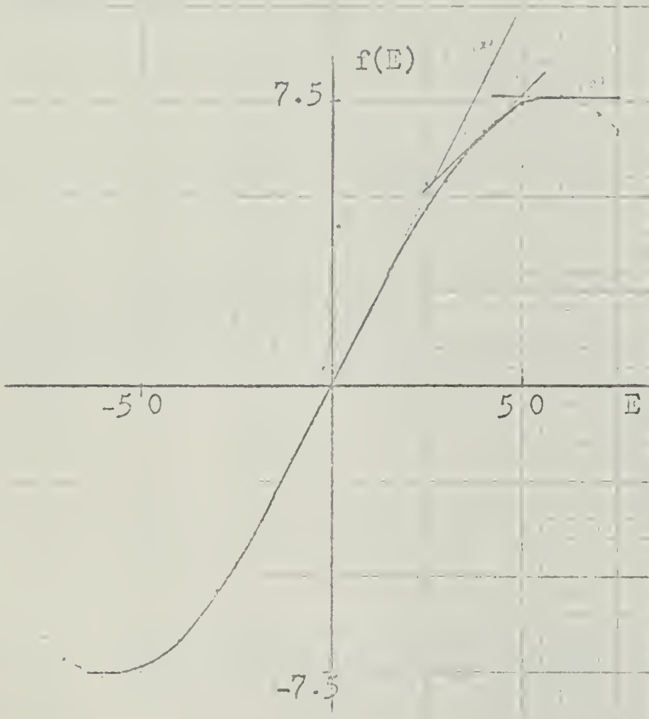


Fig.5-3 Characteristic Curve of
$f(E) = 2E - 0.020E^3$ From
Calculation

For the simulation of the restoring force function of nonlinearity from the Computer, it is approximated by three straight lines, one of which with a slope of 2; one of which with a slope of 1; and the other with a slope of zero. The circuit setup in the computer is shown in Fig. 5-4, all components used are schematically in the diagram.

The characteristic curve of restoring force function from the computer is shown in Fig. 5-5; (a) shows the waveforms of the input and the output; (b) shows the characteristic curve of the nonlinearity.

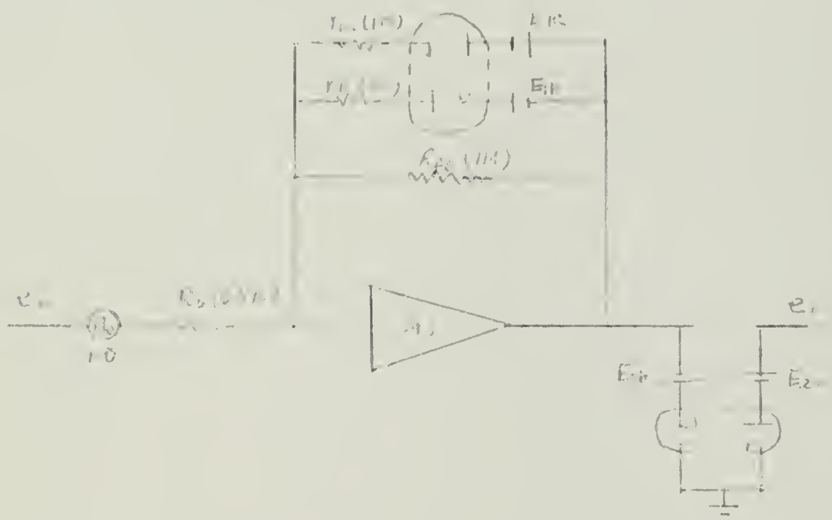
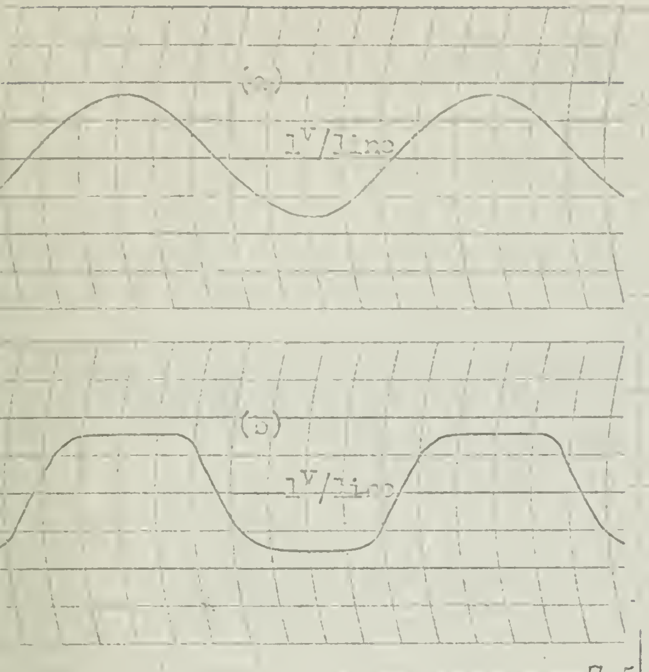


Fig. 5-4 Analog Computer Setup for
$f(E) = 2E - 0.020E^3$ Simulation

The operational block diagram and the operational circuit diagram are shown in Fig. 5-6 and Fig. 5-7. The all components used in the operational circuit diagram are sechematically in the diagram.

The results from the computer are condensed in Table 5-1, and with response curves shown in the E vs ω plane and θ_i vs ω plane are shown in Fig. 5-8 and Fig. 5-9 respectively.



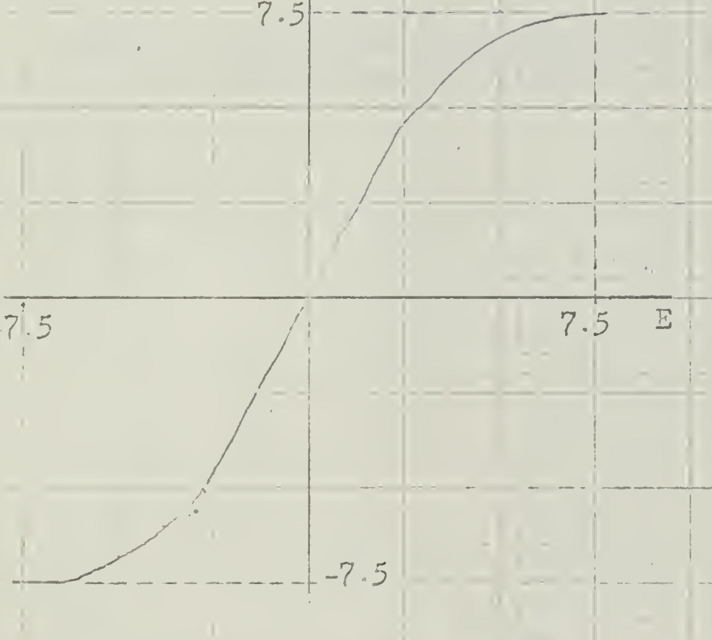


Fig.5 5 Characteristic Curve of
$f(E) - 2E = 0.020E^3$ From
The Computer

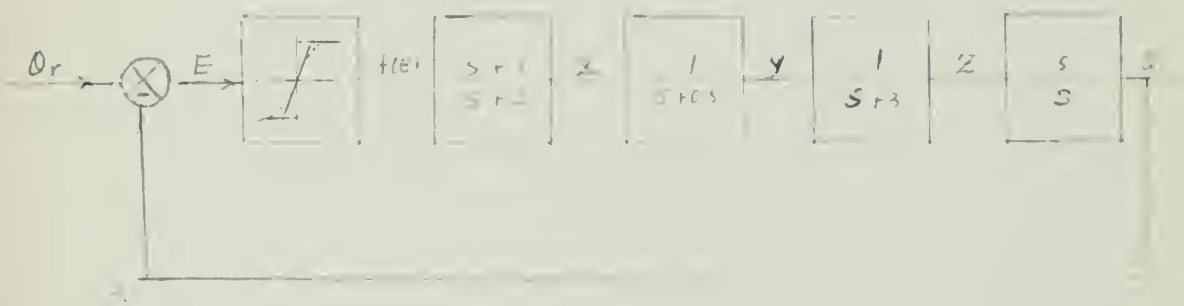


Fig. 5-6 Operational Block Diagram setup in the Analog Computer

Table 5-1. Data for Response Curve from Computer for a Higher Order System; in which $f(E) = 2E - 0.020E^3$, $\theta_i(t) = 3.0 \cos(\omega t + \theta)$

(a) Frequency Increasing:

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (ϕ)	Amplitude of Output (C)	Phase of Output (ϕ)
0.314	0.40	270.00	3.10	0.00
0.377	0.55	270.00	3.30	0.00
0.440	0.75	262.00	3.40	0.00
0.503	0.90	255.00	3.50	5.80
0.565	1.10	246.00	3.60	9.00
0.628	1.25	241.00	3.80	12.50
0.755	1.70	235.00	4.30	20.00
0.880	2.50	220.00	5.00	24.50
1.000	7.40	105.00	6.30	107.50
1.130	6.90	64.00	5.10	120.00
1.260	6.50	37.50	4.25	137.50
1.380	6.10	25.00	3.50	155.00
1.510	5.60	15.00	3.00	165.00
1.640	5.25	10.00	2.50	172.00
1.760	4.90	7.50	2.00	180.00
1.890	4.50	5.00	1.70	186.00
2.200	4.00	0.00	1.20	195.00
2.500	3.50	0.00	0.85	210.00
2.810	3.30	0.00	0.60	220.00
3.140	3.20	0.00	0.40	232.00
3.460	3.10	0.00	0.30	245.00
3.770	3.00	0.00	0.25	260.00
4.090	3.00	0.00	0.20	270.00

(b) Frequency Decreasing

Frequency of Input (ω)	Amplitude of Error Signal (E_1)	Phase of Error (ϕ)	Amplitude of Output (C)	Phase of Output (ϕ)
2.510	3.50	0.00	0.78	210.00
2.200	4.00	0.00	1.20	195.00
1.890	4.50	5.00	1.65	185.00
1.760	4.90	7.50	2.00	180.00
1.640	5.25	10.00	2.50	171.00
1.510	5.70	15.00	3.00	162.00
1.380	6.10	25.00	3.50	155.00
1.260	6.60	37.50	4.25	137.00
1.130	6.60	64.00	5.00	121.00
1.000	7.40	105.00	6.25	106.00
0.943	/	/	6.80	78.00
0.880	6.70	150.00	6.00	25.00
0.754	1.80	235.00	4.30	20.00
0.628	1.25	248.00	3.80	16.00

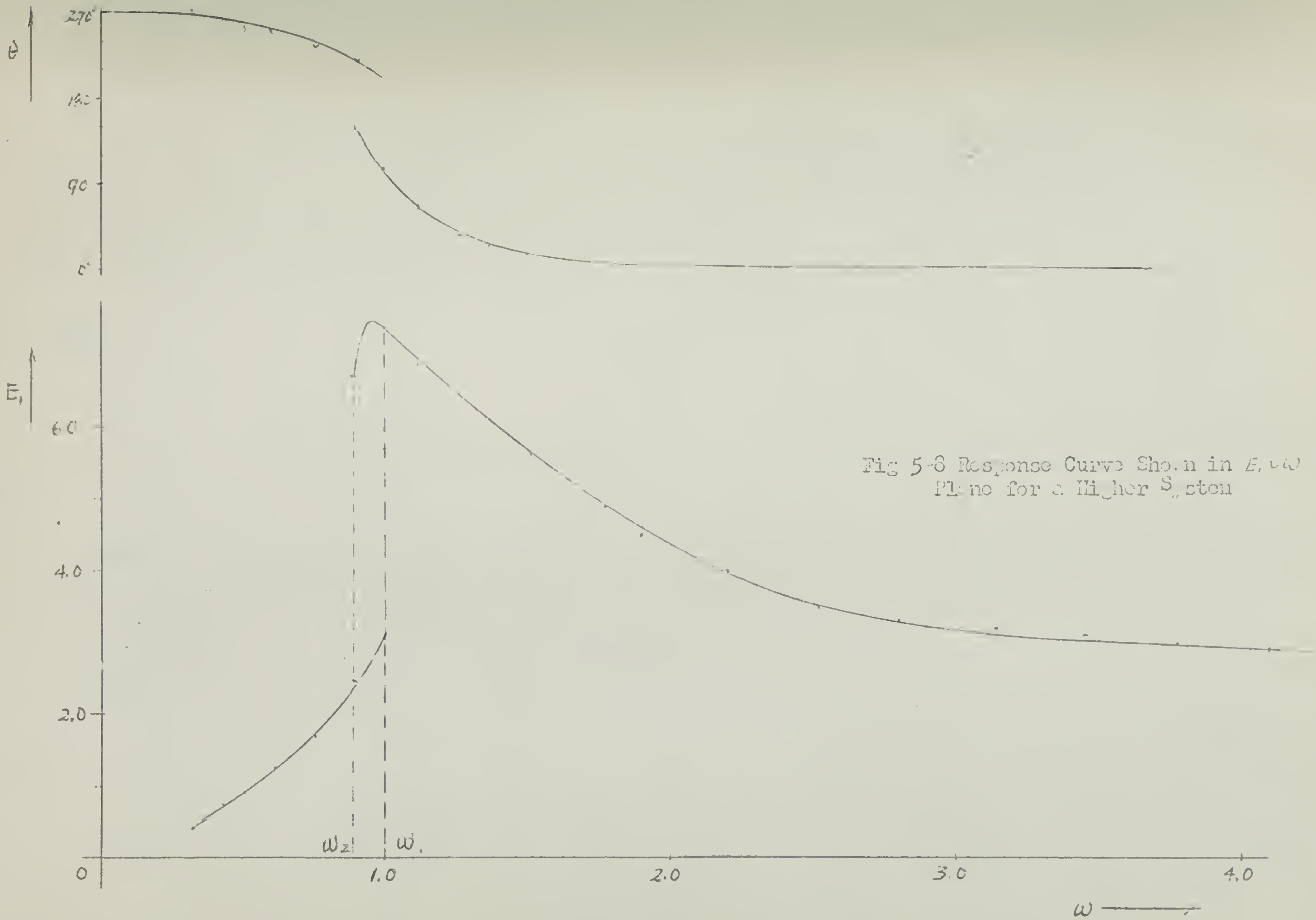


Fig 5-8 Response Curve Shown in E, ω Plane for a Higher System

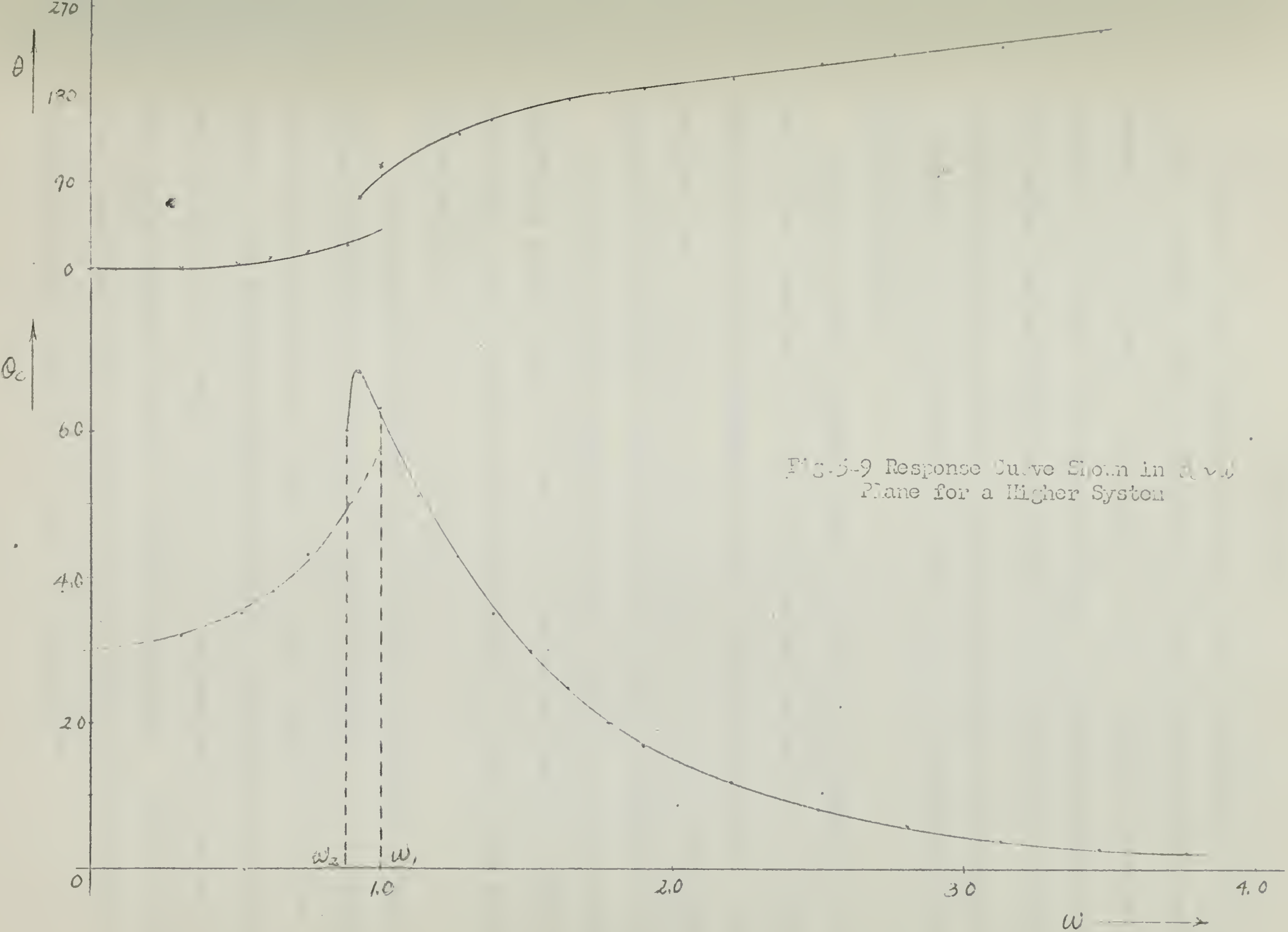


Fig. 5-9 Response Curve Shown in θ, ω Plane for a Higher System

CHAPTER VI

CONCLUSIONS

The aim of this dissertation was to present a method for investigating the conditions for existence of the forced oscillations of a nonlinear feedback control system. This method is called "harmonic linearization method", it is based on the harmonic balance and iteration method. From this method, the conditions for existence of the fundamental frequency and the sub-harmonic of order 2 and 3 are investigated, and also a equivalent gain of the nonlinearity in terms of input amplitude and frequency has been developed.

The term of the "response curve" as used here is defined as the combination of two components, one is the attenuation frequency; it is either shown in the "Error vs Frequency" plane or "Output vs Frequency" plane, and the other is the phase frequency response, it is with a difference value from the different representation of attenuation frequency response.

The results of this analysis show that a nonlinear feedback control system may possess more than one type of forced oscillations, which type will exist, it is depending on the characteristic of the system and the characteristic of input forcing function.

For any nonlinear feedback control system, the response curve of either fundamental or sub-harmonic forced oscillations, it is usually a sudden jump in amplitude and phase when a jump transient take place. The range of jump frequency depends on the value of damping of the system and the amplitude of the input; usually it is increased as the damping decreasing, and decreased as the input amplitude decreasing. A vertical tangents locus investigating for calculating the jump frequencies in terms of system constants and the amplitude of input was investigated.

Due to the resonance phenomena and the linear approximations of restoring force function, there is some difference between the theoretical and the experimental response curves. It is only existant in the jump frequency range, other than this range of frequencies, it is very close to each other.

Sub-harmonic oscillations are undesired for a nonlinear feedback control system, it is actually an unstable phenomenon. Under suitable conditions, a multiple order sub-harmonic forced oscillation may exist in a system with a constant input forcing function. There is also a jump phenomena for each order of sub-harmonic oscillations. The conditions for the existence of the 2nd and 3rd order sub-harmonic forced oscillations are investigated, and also theoretical response curve equations for each case are developed.

It is possible an important advantage that the design and analysis of a nonlinear feedback control system by using the harmonic linearization method, by means for investigating an equivalent gain of the nonlinearity. In this case, all principles and techniques applied for linear system are applicable.

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